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A SURVEY OF RESULTS CONCERNING GENERALIZED CONTINUITY ON TOPOLOGICAL SPACES

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Introduction

The notion of continuity is one of the most important tools of whole mathematics.

Starting from early years of modern mathematics, many different types of almost continuity notions were introduced, e.g., lower (upper) semi-continuity, Baire classes of functions or functions which have the Baire property, etc. Today, they are "normal and nice."

Here, we will especially investigate the following classes of almost continuous functions: quasi-continuous, somewhat continuous, nearly continuous and somewhat nearly continuous.

Quasi-continuity was introduced in 1932, by S. Kempisty [12]. He considered this class while extending some classical results of H. Hahn and R. Baire concerning separately continuous real-valued functions of many real variables. Also A. Alexiewicz and W. Orlicz [1] used quasi-continuous functions investigating some function spaces.

Somewhat continuous functions, defined by Z. Frolik [7], arise naturally while studying the invariance of Baire spaces under mappings, see also [18].

Every mathematician agrees on the importance of Banach's Closed Graph Theorem. But observe, this "almost continuity" hypothesis in this theorem is just, near continuity (see [3], thm 4 p. 40)!

Due to the fact that somewhat nearly continuous functions generalize simultaneously two important classes, namely near continuous and quasicontinuous, it is hoped that somewhat near continuity may be used in both separate versus joint continuity problems as well as in the Closed Graph Theorem.

The numeration of the results is continuous. To preserve the completeness of the reasonings we quote the published results or well-known facts. They are denoted here only by numbers (without the word Theorem or Lemma, etc.). In contrast to this, new results are called Theorems, Lemmas, Examples, etc.

Basic definitions

A *space* means a topological space. All kinds of spaces related to the compactness (like locally compact spaces etc.) are assumed to be Hausdorff.

A subset A of a space X is said to be *semi-open* if there exists an open set U in X such that $U \subset A \subset \mathrm{Cl}\, U$.

A subset B of a space X is said to be *nearly-open* if it is contained in the interior of its closure. A subset C is *somewhat nearly-open* if the interior of its closure is not empty.

II. Functions.

Functions are not necessarily assumed to be continuous.

Let $f\colon X \to Y$ be a function. We say that f is

(i) *quasi-continuous* if inverse image of every open set is semi-open, or, which is equivalent, see [2] and [20], if for every point $x \in X$ and every open sets U and V containing x and $f(x)$, respectively, $U \cap \mathrm{Int} f^{-1}(V) \neq \emptyset$.

(ii) *somewhat-continuous* if inverse image of every open set, if not empty, has a non-empty interior [7].

(iii) *nearly continuous*, if inverse image of every open set is nearly-open, or, which is equivalent, if for every point $x \in X$ and for every open set V containing $f(x)$, the point x is in the interior of the closure of $f^{-1}(V)$ [27].

(iv) *somewhat nearly continuous*, if inverse image of every open set, if not empty, is somewhat nearly-open.

The following Diagram 1 illustrates the relations between these classes of functions ($\Rightarrow$ denotes the inclusion). None of these implications can, in general, be replaced by an equivalence. The examples showing this are not difficult and the reader can construct them easily.

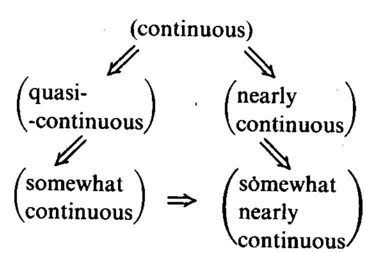

Diagram 1

Notation

The closure of a set A we denote by Cl A, the interior by Int A. We use the standard set-theoretical notation. For a given set X we use many times some families of subsets of X. Let us write them together.
Namely:

S.O.(X) denotes the family of all semi-open subsets of X.
W.I.(X) denotes the family of all subsets of X having non-empty interior relative to X.
N.O.(X) denotes the family of all nearly-open subsets of X.
S.N.O.(X) denotes the family of all somewhat nearly open subsets of X.

As remarked at the beginning of Introduction. a.c. function stands for almost continuous function. The letters N, Z, Q and R stand for the set of natural, integer, rational and real numbers, respectively. The letter I stands for the unit interval $\langle 0, 1 \rangle$. For a given function f, the set of points of continuity of f is denoted by $C(f)$. The open ball with the centre p and the radius q is denoted by $S(p, q)$.

Comprehensive investigations of some classes of almost continuous functions

§ 1. On S.O.(X), W.I.(X), N.O.(X) and S.N.O.(X)

In this paragraph we state some technical lemmas; all of them will be used in the sequel.

Lemma 1.1. Let X be a space. If D is dense in X, $A \in$ S.O.(X), then $A \cap D \in$ - S.O.(D).

Proof. A is semi-open in X, i.e., there is an open set $G \subset X$ such that

$$G \subset A \subset \mathrm{Cl}_X G$$

Now we have

$$G \cap D \subset A \cap D \subset \mathrm{Cl}_X G \cap D = \mathrm{Cl}_D G \cap D = \mathrm{Cl}_D (G \cap D)$$

and thus $A \cap D$ is semi-open in D. □

Remarks 1.2. One may show that Lemma 1.1 holds also for W.I.(X).

For some similar results to the following lemmas, however concerning S.O.(X), the reader is referred to [24].

Proofs of these lemmas are rather standard and thus they can be omitted.

Lemma 1.3. If A is a non-empty nearly-open set, then $\mathrm{Int}\,\mathrm{Cl}\,A \neq \emptyset$, i.e. A is somewhat nearly-open.

(1.4.) ([13], § 16, I, p. 151. 152). Let $\{X_t \mid tT\}$ be any family of spaces and let ΠA_t be a subset of the Cartesian product ΠX_t of X_t.

Then:
(1) $\operatorname{Int} \Pi A_t = \Pi \operatorname{Int} A_t$, if $A_t = X_t$ except for a finite number of t's.
(2) $\operatorname{Cl} \Pi A_t = \Pi \operatorname{Cl} A_t$

Lemma 1.5. Let $\{A_t \mid t \in T\}$ be a collection of

$$\left\{\begin{array}{l}\text{sets with the non-empty interiors}\\ \text{nearly-open sets}\\ \text{somewhat nearly-open set}\end{array}\right\} \text{ in } X. \text{ Then } \bigcup_{t \in T} A_t$$

$$\text{is a } \left\{\begin{array}{l}\text{set with the non-empty interior}\\ \text{nearly-open set}\\ \text{somewhat nearly-open set}\end{array}\right\} \text{ in } X.$$

Lemma 1.6. Let $\{X_t \mid t \in T\}$ be any family of spaces X_t, let $X = \Pi X_t$ and $\prod_{j=1}^{n} A_{t_j} \times \prod_{t \neq t_j} X_t$ a non-empty subset of X, where $n \in N$.

$$\text{Then } \left\{\begin{array}{l}A_{t_j} \in \text{W.I.}(X_{t_j})\\ A_{t_j} \in \text{N.O.}(X_{t_j})\\ A_{t_j} \in \text{S.N.O.}(X_{t_j})\end{array}\right\} \text{ for each } j \ (1 \leq j \leq n) \text{ if}$$

$$\text{and only if } \left\{\begin{array}{l}A \in \text{W.I.}(X)\\ A \in \text{N.O.}(X)\\ A \in \text{S.N.O.}(X)\end{array}\right\}.$$

Proof. It easily holds if we recall that by (1.4)

$$\operatorname{Int} \operatorname{Cl} A = \prod_{j=1}^{n} \operatorname{Int} \operatorname{Cl} A_{t_j} \times \prod_{t \neq t_j} X_t.$$

§ 2 Classes of functions

In this paragraph we will investigate the behaviour of some types of a.c. functions under the operations like the composition, the graph-function, the generalized product, some special restrictions, taking limits of some sequences of them, invariance of them under some topological properties. We will also prove some special properties of these functions. Each time we recall then the behaviour of a continuous function.

Obviously, it is true that:
(2.1) The composition of two continuous functions is a continuous function. It is proved in [14], Remark 12, p. 40.

(2.2) The composition of two quasi-continuous functions need not be quasi-continuous.

Using the same functions as in [14], Remark 12, p. 40 one can verify the following

Propositione 2.3.

The composition of two $\left\{\begin{matrix}\text{somewhat continuous}\\ \text{somewhat nearly continuous}\end{matrix}\right\}$ functions need not be $\left\{\begin{matrix}\text{somewhat continuous}\\ \text{somewhat nearly continuous}\end{matrix}\right\}$.

A standard example shows the following

Proposition 2.4. The composition of two nearly continuous functions need not be nearly continuous.

Let A be a class of functions. If $f \circ g \in A$ implies that $f \in A$ and $g \in A$, then we say that A possesses the *factorization property of the composition.*

Not difficult example shows that

Proposition 2.5.

The class of $\left\{\begin{matrix}\text{continuous}\\ \text{quasi-continuous}\\ \text{somewhat continuous}\\ \text{nearly continuous}\\ \text{somewhat nearly continuous}\end{matrix}\right\}$,

functions does not possess the factorization property of the composition.

However, the following is true

(2.6) ([25], Thm lp. 289). If g in a somewhat continuous and somewhat open function from X onto Y, f is a function from Y onto Z, then:
f is somewhat continuous if and only if $f \circ g$ is somewhat continuous.

(2.7) ([25] Corollary p. 290) If g is a continuous and open function from X onto Y, and f is a function from Y into Z, then:
f is somewhat continuous if and only if $f \circ g$ is somewhat continuous.

Let $f\colon X \to Y$ be a function and let $g\colon X \to X \times Y$ be given by $g(x) = (x, f(x))$. We say that g is the *graph-function* of f.

It is well-known that

(2.8) A function $g\colon X \to X \times Y$ is continuous if and only if $f\colon X \to Y$ is continuous.

It is proved in [23], Theorem 2, p. 401.

(2.9) A function $g\colon X \to X \times Y$ is quasi-continuous if and only if $f\colon X \to Y$ is quasi-continuous.

Carnahan and Long proved ([5], Theorem 2, p. 414)

(2.10) A function $g\colon X \to X \times Y$ is nearly-continuous if and only if $f\colon X \to Y$ is nearly-continuous.

One may conjecture that if f is $\left\{\begin{matrix}\text{somewhat continuous} \\ \text{somewhat nearly continuous}\end{matrix}\right\}$, then g is $\left\{\begin{matrix}\text{somewhat continuous} \\ \text{somewhat nearly continuous}\end{matrix}\right\}$.
It is not so even in the case when $X = Y = R$, f being a bijection.

Example 2.11. We make use of the example of a somewhat continuous bijection which is not quasi-continuous ([18], Proposition 1, p. 174). Let f: $R \to R$ be defined as follows: $f(x) = x$, if $x \neq 0$, $x \neq 1$; $f(0) = 1$, $f(1) = 0$. Take the open ball $S\left((0, 1), \frac{1}{2}\right)$. We see that $g^{-1}\left(S\left((0, 1], \frac{1}{2}\right)\right)$ is non-empty. However, $\operatorname{Int} g^{-1}\left(S\left((0,1), \frac{1}{2}\right)\right) = \emptyset$.

Let A be a class of functions. If g: $X \to X \times Y$ given by $g(x) = (x, f(x))$ belongs to A and this implies that f: $X \to Y$ belongs to A, then we say that A possesses *the factorization property of the graph-function.*

It follows from (2.8)–(2.10) that the classes of continuous, quasi-continuous and nearly-continuous functions have the factorization property of the graph-function.

In [2], Theorem 2.8, p. 320, a stronger result is proved than the following

(2.12) If the graph-function g of a function f is somewhat continuous, then f is somewhat continuous.

Proposition 2.13. If the graph-function g of a function f is somewhat nearly continuous, then f is somewhat nearly continous.

Proof. Let $V \subset Y$ be an open set with $f^{-1}(V) \neq \emptyset$ and since g is somewhat nearly continuous, $\emptyset \neq \operatorname{Int}\operatorname{Cl} g^{-1}(X \times V) = \operatorname{Int}\operatorname{Cl}(X \cap f^{-1}(V)) = \operatorname{Int}\operatorname{Cl} f^{-1}(V)$. Then f is somewhat nearly continuous.

Let $\{X_t \mid t \in T\}$ and $\{Y_t' \mid t \in T\}$ be any two families of spaces with the same index set T. For each $t \in T$, let f_t: $X_t \to Y_t$ be a function. Let a function f: $\Pi X_t \to \Pi Y_t$ defined by $f((x_t)) = f_t(x_t)$ be given. The function f is called *the product of functions*. Let A be a class of functions. If f: $\Pi X_t \to \Pi Y_t$ defined by $f((x_t)) = f_t(x_t)$ belongs to A and this implies that f_t, for all $t \in T$, belongs to A, then we say that A possesses the *factorization property of the product of functions*, or shortly, the factorization property of the product.

It is well-known that

(2.14) A function f: $\Pi X_t \to \Pi Y_t$ is continuous if and only if f_t is continuous, for each $t \in T$.

In [24], Theorem 5, p. 135, it is proved that

(2.15) A function f: $\Pi X_t \to \Pi Y_t$ is quasi-continuous if and only if f_t is quasi-continuous, for each $t \in T$.

Proposition 2.16. A function f: $\Pi X_t \to \Pi Y_t$ is

$\left\{\begin{array}{l}\text{somewhat continuous}\\ \text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\}$ if and only if f_t is

$\left\{\begin{array}{l}\text{somewhat continuous}\\ \text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\}$, for each $t \in T$.

Proof. We will prove Proposition 2.16 for the class of nearly continuous functions. Proofs of the other cases are similar.

Sufficiency. Let V be an open set in ΠY_t. Then, there are $t_j \in T (1 \leq j \leq n)$ and open sets V_{t_j} in Y_{t_j} such that $V = \prod_{j=1}^{n} \times \prod_{t \neq t_j} Y_t$. Since f_{t_j} is nearly continuous, $f_{t_j}^{-1}(V_{t_j})$ is nearly-open in X_{t_j} for each j $(1 \leq j \leq n)$. If there exists t_j such that $f_{t_j}(V_{t_j}) = \emptyset$, then $f^{-1}(V) = \prod_{j=1}^{n} f_{t_j}^{-1}(V_{t_j}) \times \prod_{t \neq t_j} X_t = \emptyset$. Hence $f^{-1}(V)$ is nearly-open in ΠX_t.

If $f_{t_j}^{-1}(V_{t_j}) \neq \emptyset$ for all j $(1 \leq j \leq n)$, then $\prod_{j=1}^{n} f_{t_j}^{-1}(V_{t_j}) \times \prod_{t \neq t_j} X_t \neq \emptyset$. Hence by Lemma 1.6., $f^{-1}(V) = \prod_{j=1}^{n} f_{t_j}^{-1}(V_{t_j}) \times \prod_{t \neq t_j} X_t$ is nearly-open in ΠX_t. Now, for any open set W in Y there exists a family $\{V_\beta | \beta \in B\}$ of open sets such that $W =$
$= \prod_{\beta \in B} f^{-1}(V_\beta)$. Hence by Lemma 1.5. $f^{-1}(W) = \bigcup_{\beta \neq B} f^{-1}(V_{t\beta})$ is nearly-open in ΠX_t. This implies that f is nearly-continuous.

Necessity. For each fixed $t \in T$, let p_t: $\prod_{\alpha \in t} Y_\alpha \rightarrow Y_t$ be the projection. Suppose V_t is an arbitrary open set in Y_t. Then $p_t^{-1}(V_t) = V_t \times \prod_{\alpha \neq t} Y_\alpha$ is open in ΠY_α. Since f is nearly-continuous, $f^{-1}(p_t^{-1}(V)) = f_t^{-1}(V_t) \times \prod_{\alpha \neq t} X_\alpha$ is nearly-open in ΠX_α. If $f_t^{-1}(V_t)$ is empty, then it is obvious that f_t is nearly-continuous. If $f_t^{-1}(V_t)$ is not empty, then $f_t^{-1}(V_t) \times \prod_{\alpha \neq t} X_\alpha \neq \emptyset$ and hence by Lemma 1.6., $f_t^{-1}(V_t)$ is nearly-open in X_t. This implies that f_t is nearly-continuous. □

Now, we will investigate the behaviour of a.c. functions under certain types of restrictions.

It is well known that:

(2.17) Arbitrary restrictions of a continuous function are continuous.

It is proved in [24], Theorem 3, p. 134, that:

(2.18) The restriction of a quasi-continuous function to an open subspace is quasi-continuous.

(2.19) ([15] Theorem 4, p. 177) The restriction of a nearly-continuous function to an open subspace is nearly-continuous.

(2.20) ([2] Example 2.6., p. 319) The restriction of a somewhat continuous function to an open subspace need not be somewhat continuous.

Proposition 2.21. The restriction of a somewhat nearly continuous function to an open subspace need not be somewhat continuous.

Proof. Take the example of a somewhat continuous bijection which is not quasi-continuous, [18], Proposition 1, p. 174.

Then $G = \left(-\frac{1}{2}, \frac{1}{2}\right)$ is the open subspace with the required property:

(2.22) ([16], [2], Example 1.10). The restriction of a quasi-continuous function to a closed subspace need not be quasi-continuous.

(2.23) ([2], Example 2.7, p. 319). The restriction of a somewhat continuous function to a closed subspace need not be somewhat continuous.

(2.24] ([15], Example 3, p. 177). The restriction of a nearly continuous function to a closed subspace need not be nearly continuous.

Using [18] Proposition 1, p. 174, one may prove the following, $\left(\text{take } \left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle \text{ as the closed subspace}\right)$.

Proposition 2.25. The restriction of a somewhat nearly continuous function to a closed subspace need not be somewhat nearly continuous.

Proposition 2.26. The restriction of a quasi-continuous function to a dense subspace is quasi-continuous.

Proof. Let D be a dense subspace of X. Let V be an open subset of Y such that $f^{-1}(V)\ D \neq \emptyset$. Since f is quasi-continuous, $f^{-1}(V)$ is semi-open in X. Now, apply Lemma 1.1.

Proposition 2.27. The restriction of a somewhat continuous function to a dense subspace is somewhat continuous.

Proof. Similar as in the proof of Proposition 2.26. Next, apply Remark 1.2.

Proposition 2.28.

The restriction of a $\left\{\begin{array}{l}\text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\}$ function to a dense subspace need not be $\left\{\begin{array}{l}\text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\}$.

Now we turn to the problem whether the limit of a sequence of a.c. functions belonging to a class A is a function from A.

Take $\{f_n\}_{n=1}^{\infty}$ defined on $\langle 0, 1\rangle$ as $f_n(x) = x^n$, for each $n = 1, 2, \ldots$. The limit function f given by $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$ is not somewhat nearly continuous though f_n is continuous, $n = 1, 2, \ldots$. So, we obtain

Proposition 2.29. The limit of a sequence of

$$\left\{\begin{array}{l}\text{continuous}\\ \text{quasi-continuous}\\ \text{somewhat continuous}\\ \text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\} \quad \text{functions need not be}$$

$$\left\{\begin{array}{l}\text{continuous}\\ \text{quasi-continuous}\\ \text{somewhat continuous}\\ \text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\}.$$

In this part (X, ϱ) denotes a separable metric space and (Y, ϱ') any metric space. The functions are defined on X and take values in Y. Let Ω be the first uncountable ordinal number. The transfinite sequence $\{a_\xi\}_{\xi < \Omega}$ of elements of a metric space with the metric ϱ' is said to be *convergent* and have a limit $a \in Y$ if for every $\varepsilon < 0$ there exists an ordinal number $\mu < \Omega$ such that for each ξ with $\mu \leq \xi < \Omega$ the inequality $\varrho'(a_\xi, a) < \varepsilon$ holds. A transfinite sequence $\{f_\xi\}_{\xi < \Omega}$ on a set X with the values in a metric space Y is said to be *pointwise convergent* to a fuction f, defined on X, if $\{f_\xi(x)\}_{\xi < \Omega}$ is convergent to $f_\xi(x)$ for any $x \in X$.

It follows from [28] Theorem 1, p. 158.

(2.30) Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of continuous functions pointwise converging to a function f. The f is continuous.

(2.31) ([21] Theorem 1, p. 110). Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of quasi-continuous functions pointwise converging to a function f. Then f is quasi-continuous.

Proposition 2.32. Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of somewhat continuous functions pointwise converging to a function f. Then f is somewhat continuous.

Proof. (Compare [21] Theorem 1, p. 110). Let f be not somewhat continuous at $x_0 \in X$. Then there is an $\varepsilon > 0$ and a $\delta > 0$ such that for any non-empty open set $G \subset S(x_1, \delta)$ there exists $f \in G$ with

$$(1) \qquad |f(t) - f(x_0)| \geq \varepsilon.$$

Hence the set T of all t for which (1) is true is dense in $S(x_1, \delta)$. Let D be a countable dense subset of T. There is $\mu < \Omega$ such that for $\xi > \mu$

$$(2) \qquad f_\xi(x) = f(x) \qquad (\text{for } x \in D).$$

The last fact easily follows from the following result of Šalát [28], p. 158.

(2.33) Let Z be a metric space, $a_\xi \in Z(\xi < \Omega)$ and $a_\xi \to a$. Then there exists an ordinal number $\alpha < \Omega$ such that $a_\xi = a$ for each ξ, $\alpha \leq \xi < \Omega$.

Now, let $\xi_0 > \mu$ be any fixed ordinal number. The somewhat continuity of f_{ξ_0} at x_0 implies the existence of a non-empty open set $U \subset S(x_1, \delta)$ such that $|f_{\xi_0}(x) - f_{\xi_0}(x_0)| < \varepsilon$ for $x \in U$. Evidently, $U \cap D \neq \emptyset$, D being dense in $S(x_1, \delta)$. For any $t \in U \cap D$ we have $|f_{\xi_0}(x_0) - f_{\xi_0}(t)\} < \varepsilon$. In view of (2.33) we may assume $f_{\xi_0}(x_0) = f(x_0)$, $f_{\xi_0}(t) = f(t)$ (in view of (2)), hence $|f(x_0) = f(t)| < \varepsilon$, which is a contradiction to (1). This finishes the proof of Proposition 2.32. □

(2.34) ([22], Theorem 3, p. 124). Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of nearly continuous functions pointwise converging to a function f. Then f need not be nearly continuous.

Proposition 2.35. Let $\{f_\xi\}$ $(\xi < \Omega)$ be a transfinite sequence of somewhat nearly continuous functions pointwise converging to a function f. Then f need not be sumewhat nearly continuous.

Proof. One may verify that the function f in [22], Theorem 3, p. 124, is not somewhat nearly continuous, though $f_\xi(\xi < \Omega)$ are somewhat nearly continuous. □

Now we turn to the uniform convergence of functions.

The following result is well-known:

(2.36) Let X be a space (Y, ϱ), be metric and let $\{f_i\}$ be a sequence of continuous functions from X to Y. If the sequence $\{f_i\}$ is uniformly convergent to a function f, then f is continuous.

Proposition 2.37. Let X be a space, (Y, ϱ) be metric and let $\{f_i\}$ be a sequence of {quasi-continuous / somewhat continuous / nearly continuous / somewhat nearly continuous} functions from X to Y. If the sequence $\{f_i\}$ is uniformly convergent to a function f, then f is {quasi-continuous / somewhat continuous / nearly continuous / somewhat nearly continuous}.

Proof. We will prove Proposition 2.37 for the class of somewhat continuous functions. The proofs for the other cases are similar.

In fact, we will show that for every $x_0 \in X$ and any $\varepsilon > 0$ there exists an open, non-empty set U such that $\varrho(f(x_0), f(x_1)) < \varepsilon$, for every $x_1 \in U$.

Let us take k such that

$$(*)\ \varrho(f(x), f_n(x)) < \frac{\varepsilon}{3}, \text{ for every } x \in X \text{ and } n \geq k.$$

Since the function f_k is somewhat continuous, there is an open, non-empty set U with

(**) $\varrho(f_k(x_0), f_k(x_1)) < \frac{\varepsilon}{3}$, for every $x_1 \in U$.

Now we will prove that the set U has the required property. Let us take $x_1 \in U$. It follows form (*) and (**) that

$$\varrho(f(x_0), f(x_1)) \leq \varrho(f(x_0), f_k(x_0)) + \varrho(f_k(x_0), f_k(x_1)) + \\ + \varrho(f_k(x_1), f(x_1)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \square$$

The most of the results of § 2 obtained up to now are summarized in the following *Table 1.*

Table 1

Functions	The composition property	The factorization property of the composition	The graph-function	The factorization property of the graph-function	The product of functions	The factorization property of the product of functions	Restrictions to the open subspaces	Restrictions to the closed subspaces	Restrictions to the dense subspaces	Limit of a sequence of functions	Limit of a transfinite sequence of functions	Limit of uniformly conver- -engent sequence of functions
continuous functions	+ (2.1)	− (2.5)	+ (2.8)	+ (2.8]	+ (2.14)	+ (2.14)	+ (2.17)	+ (2.17)	+ (2.17)	− (2.29)	+ (2.30)	+ (2.36)
quasi- -continuous functions	− (2.2)	− (2.5)	+ (2.9)	+ (2.9)	+ (2.15)	+ (2.15)	+ (2.18)	− (2.22)	+ (2.26)	− (2.29)	+ (2.31)	+ (2.37)
somewhat continuous functions	− (2.3)	− (2.5)	− (2.11)	+ (2.12)	+ (2.16)	+ (2.16)	− (2.20)	− (2.23)	+ (2.27)	− (2.29)	+ (2.32)	+ (2.37)
nearly continuous functions	− (2.4)	− (2.5)	+ (2.10)	+ (2.10)	+ (2.16)	+ (2.16)	+ (2.19)	− (2.24)	− (2.28)	− (2.29)	− (2.34)	+ (2.37)
somewhat nearly continuous functions	− (2.3)	− (2.5)	− (2.11)	+ (2.13)	+ (2.16)	+ (2.16)	− (2.21)	− (2.25)	− (2.28)	− (2.29)	− (2.35)	+ (2.37)

Now we turn to the investigations of the invariants and special properties of considered a.c. functions.

It is well known that

(2.38) Separability, connectedness and compactness are the invariants under continuity.

Proposition 2.39. Connectedness or compactness are not the invariants under

$$\left\{\begin{array}{l}\text{quasi-continuity}\\ \text{somewhat continuity}\\ \text{near continuity}\\ \text{somewhat near continuity}\end{array}\right\}$$

(2.40) ([2], Theorem 2.9, p. 320). Separability is an invariant under somewhat continuity (thus under quasi-continuity).

Proposition 2.41. Separability is not an invariant under near continuity (thus under somewhat near continuity).

If F is a linear subspace of a linear topological space X, the linear topological space is the quotient space X/F with the topology such that the set U in X/F is open if and only if $Q^{-1}(U)$ is open in X, where $Q: X \to X/F$ is the quotient functions; that is $Q(x) = x + F$. For all notions used but undefined here see [11], pp. 9, 34 and 39.

(2.42) ([11], Theorem 5.7. p. 39). Let F be a linear topological subspace of a linear topological space X and let X/F be the quotient space. Then the quotient function $Q: X \to X/F$ is linear, continuous and open.

The following result which is a consequence of (2.42) and (2.7), generalizes the second part of Theorem 5.7 of [11].

(2.43) [2.5], Thm 3p. 290). A function T on X/F is somewhat continuous if and only if the composition $T \circ Q$ is somewhat continuous.

(2.44] ([25], Thm 4p. 290). If T is a somewhat continuous linear transformation from a linear topological space X to a linear topological space Y, then T is continuous.

(2.45) ([25], Thm 4p. 290). If f is a somewhat continuous functional on a linear topological space, then f is continuous.

Corollary 2.46. If T is a quasi-continuous linear transformation from a linear topological space X to a linear topological space Y, then T is continuous.

(2.47) (cf. [10], p. 214). If T is a nearly continuous (thus somewhat nearly continuous) linear transformation from a linear topological space X to a linear topological space Y, then T need not be continuous.

However, if we assume that T has the closed graph and X and Y are "nice" linear topological spaces, then T is continuous — see [30].

The following result may be found in [8], p. 11.

(2.48) If f is a somewhat continuous (thus quasi-continuous) homomorphism between topological groups, then f is continuous.

Modifying the example in [10], Problem R p. 214, we obtain

(2.49) If f is a nearly continuous (thus somewhat nearly continuous) homomorphism between topological groups, then f need not be continuous.

Let X be a space and Y be a Hausdorff space. It is well-known that if a continuous function f of a dense subset D of a space X to Y is continuously extendable over X, then the extension is uniquely determined by f.

Proposition 2.50. If a $\left\{\begin{array}{l}\text{quasi-continuous}\\ \text{somewhat continuous}\\ \text{nearly continuous}\\ \text{somewhat nearly continuous}\end{array}\right\}$ function f of a dense subset D of space X to Y is $\left\{\begin{array}{l}\text{quasi-continuously}\\ \text{somewhat continuously}\\ \text{near continuously}\\ \text{somewhat near continuously}\end{array}\right\}$ extendable over X, then the extension is not uniquely determined by f.

Now we summarize the results of § 2 concerning the discussed invariants and special properties of a.c. functions — see

Table 2

Functions	Preservation of separability	Preservation of connectedness	Preservation of compactness	Imply continuity of linear transformations	Imply continuity of homorphism of topological groups	The extension from a dense subset is uniquely determined
continuous functions	+ (2.38)	+ (2.38)	+ (2.38)	+	+	+
quasi-continuous functions	+ (2.40)	– (2.39)	– [2.39)	+ (2.46)	+ (2.48)	– (2.50)
somewhat continuous functions	+ (2.40)	– (2.39)	– (2.39)	+ (2.44)	+ (2.48)	– (2.50)
nearly continuous functions	– (2.41)	– (2.39)	– (2.39)	– (2.47)	– (2.49)	– (2.50)
somewhat nearly continuous functions	– (2.41)	– (2.39)	– (2.39)	– (2.47)	– (2.49)	– (2.50)

§ 3 Symmetric quasi-contunity

At the beginning we recall the definitions of notion used in this paragraph.

A space will be called a *Baire space* [4], if its non-empty open sets are of second category.

A function $f\colon X \times Y \to Z$ (X, Y, Z — arbitrary topological spaces) is said to be *quasi-continouous at* $(p, q) \in X \times Y$ *with respect to the variable* y, (compare [16], p. 41 also [12], p. 188), if for every neighborhood N of $(f(p, q)$ and for every neighborhood $U \times V$ of (p, q), there exists a neighborhood V' of q, with $V' \subset V$, and a non-empty open $U' \subset U$, such that for all $((x, y) \in U' \times V'$ we have $f(x, y) \in N$. If f is quasi continuous with respect to the variable y at each point of its domain, it will be called *quasi-continuous with respect to* y. The definition of a function f that is quasi-continuous with respect to x is quite similar. If f is quasi-continuous with respect to x and y, we will say that f is *symmetrically quasi-continuous*.

Let X, Y and Z be spaces and let a function f: $X \times Y \to Z$ be given. For every fixed $x \in X$, the function f: $X \times Y \to Z$ defined by $f_x(y) = f(x, y)$, where $y \in Y$, is called an *x-section of f*. A y-section of f is defined similarly.

Let A be a class of functions. We say that a function f: $X \times Y \to Z$ is *separately of class A* (e. g., separately continuous), if f is of class A (resp. continuous) with respect to each variable while the other variable is fixed. So, f is separately of class A, if its all x-sections f_x and y-sections f_y are of class A.

To express that a function f: $X \times Y \to Z$ is of class A as the function of two variables, we will say that f is *jointly* of class A (e.g. jointly continuous).

One can easily show from the definitions that if f is symmetrically quasi-continuous, then f_x and f_y are quasi-continuous for all $x \in X$ and $y \in Y$. The converse does not hold.

(3.1) ([25], Ex. 1 p. 349). Indeed, define f: $J \times J \to J$ as follows: $f(x, y) =$ $=$ if $x \in \left\langle \frac{1}{2}, 1 \right\rangle$ and $y \in \left\langle 0, \frac{1}{2} \right\rangle$ and $f(x, y) = 1$ on the rest. It is easy to verify that all x-sections f_x and all y-sections f_y of f are quasi-continuous and f is not symmetrically quasi-continuous. However, we have the following:

(3.2) ([26], Thm 1 p. 350). Let X be a Baire space, Y be first countable and Z be regular. If f is a function on $X \times Y$ to Z such that all its x-sections f_x are continuous and all its y-sections f_y are quasi-continuous, then f is quasi-continuous with respect to y.

As an immediate consequence we obtain (3.3) ([26], Cor. 1 p. 350). Let X and Y be first countable, Baire spaces and Z be a regular one. If f: $X \times Y \to Z$ is separately continuous, then f is symmetrically quasi-continuous.

(3.4) ([26], Remark 1 p. 351). The assumption of quasi-continuity of f_y cannot be weakened. There is a counterexample that using somewhat continuity instead of quasi-continuity, Theorem 3.2. becomes false. Indeed, define f: $J \times J \to J$ by: $f(x, y) = 0$ if $x \in \left\langle 0, \frac{1}{4} \right\rangle$ or $x \in \left(\frac{1}{2}, 1 \right\rangle \cap Q$; $f(x, y) = 1$ if $x \in \left(\frac{1}{4}, \frac{1}{2} \right\rangle$ or $x \in \left(\frac{1}{2}, 1 \right\rangle \cap (R \setminus Q)$.

Such a function has all its x-sections continuous and has all its y-sections somewhat continuous and Theorem 3.2. does not hold for f. Also the assumption of continuity of all x-section f_x of f is essential — see example 3.1.

Now let us turn to symmetrically quasi-continuous functions and their connections with separately continuous functions.

A theorem of Sierpiński [29] asserts that a real-valued function on R^n, if it is separately continuous, is determined by its values on any dense subset of R^n. Some abstract versions of this statement were given by Comfort [6] and Goffman and Neugebauer [9]. Symmetric quasi-continuity is closely related to the notion of separate continuity — see Corollary 3.3. One may conjecture that Sierpiński's theorem holds for symmetrically quasi-continuous functions. It is not so, even in the case when these two spaces are Euclidean ones.

Example 3.12. (Compare [16], example 2, p. 42). Let f and g be defined as follows

$$f(x, y) = \begin{cases} \sin \dfrac{1}{x^2 + y^2}, & \text{if } \; x^2 + y^2 \neq \emptyset \\ 0, & \text{if } \; x^2 + y^2 = 0 \end{cases}$$

$$g(x, y) = \begin{cases} \sin \dfrac{1}{x^2 + y^2}, & \text{if } \; x^2 + y^2 \neq 0 \\ \dfrac{1}{2}, & \text{if } \; x^2 + y^2 = 0 . \end{cases}$$

Both of the functions f and g are symmetrically quasi-continuous, they agree on the dense subset of R^2 (in fact, the whole plane, except (0, 0)); however, they are not identical.

§ 4. Comparing some a.c. functions defined on product spaces

We start from some examples. Throughout this paragraph, the square in the plane $\{(x, y) \mid -1 \leq x \leq 1, \; -1 \leq y \leq 1\}$ is denoted by S.

Example 4.1. Let f: $S \to R$ be defined as follows

$$f(x, y) = \begin{cases} 0, \text{ if } (0 \leq x \leq 1 \wedge 0 \leq y \leq 1) \vee (-1 \leq x \leq 0 \wedge -1 \leq y \leq 0) \\ 1, \text{ on the rest}. \end{cases}$$

This function is separately quasi-continuous (thus jointly quasi-continuous, by Theorem 1 of [16], p. 39). However it is neither quasi-continuous with respect to y nor jointly (nor separately) nearly continuous.

Example 4.2. Define the function f_1 and f_2 on $\langle -1, 0\rangle \times \langle -1, 0\rangle$ and $(0, 1\rangle \times \langle -1, 0)$, respectively.

Put

$$f_1(x, y) = \begin{cases} 0, \text{ if } \left(-1 \le x \le -\frac{1}{2} \wedge -1 \le y \le -\frac{1}{2}\right) \vee \\ \quad \vee \left(-\frac{1}{2} \le x \le 0 \wedge -\frac{1}{2} \le y \le 0\right) \vee \\ \quad \vee \left(x \in Q \cap \left\langle -1, -\frac{1}{2}\right\rangle \wedge y \in Q \cap \left\langle -\frac{1}{2}, 0\right\rangle\right) \vee \\ \quad \vee \left(x \in Q \cap \left\langle -\frac{1}{2}, 0\right\rangle \wedge y \in Q \cap \left\langle -1, -\frac{1}{2}\right\rangle\right) \\ 1, \text{ on the rest.} \end{cases}$$

$$f_2(x, y) = \begin{cases} 1, \text{ if } \left(0 < x \le \frac{1}{2} \wedge -1 \le y \le -\frac{1}{2}\right) \vee \left(\frac{1}{2} \le x \le 1 \wedge -\frac{1}{2} \le y < 0\right) \\ \quad \vee \left(x \in (R \setminus Q) \cap \left(0, \frac{1}{2}\right) \wedge y \in (R \setminus Q) \cap \left(-\frac{1}{2}, 0\right)\right) \vee \\ \quad \vee \left(x \in (R \setminus Q) \cap \left(\frac{1}{2}, 1\right\rangle \wedge y \in (R \setminus Q) \cap \left\langle -1, -\frac{1}{2}\right)\right) \\ 0, \text{ on the rest.} \end{cases}$$

These functions define f_3 and f_4 from $\langle 0, 1\rangle \times \langle 0, 1\rangle$ and $\langle -1, 0) \times (0, 1\rangle$, respectively, as follows

$$f_3(x, y) = f_1(-x, y + 1) \qquad f_4(x, y) = f_2(-x, y + 1).$$

Put $f(x, y) = f_i(x, y)$, where $1 \le i \le 4$. The function f is defined on S. It is easy to check that f is jointly somewhat continuous (thus jointly somewhat nearly continuous), also f is separately somewhat continuous (thus separately somewhat nearly continuous). However, f is neither jointly quasi-continuous (thus not separately quasi-continuous) nor quasi-continuous with respect to y, nor jointly (nor separately) nearly continuous.

Example 4.3. Let f: $S \to R$ be defined as follows

$$f(x, y) = \begin{cases} 1, \textit{If } (0 < x \le 1 \wedge -1 \le y \le 1) \vee (x = 0 \wedge y = 1) \\ 0, \text{ on the rest.} \end{cases}$$

This function is quasi-continuous with respect to y, however, it is not separately somewhat nearly continuous (thus not separately somewhat continuous, not separately nearly continuous). Also it is not jointly nearly continuous.

Example 4.4. Let f: $S \to R$ be defined as follows

$$f(x, y) = \begin{cases} 0, \text{ if } \left(-1 \leq x \leq -\frac{1}{2} \wedge -1 \leq y \leq 1\right) \vee (x = 0 \wedge y \neq 0) \\ \qquad \vee \left(x \in \left\langle -\frac{1}{2}, \frac{1}{2}\right) \cap Q \setminus \{0\} \wedge y \in \langle -1, 1 > \cap Q\right) \\ 1, \text{ on the rest.} \end{cases}$$

The function f is jointly somewhat continuous. However it is neither separately somewhat nearly continuous (thus not separately somewhat continuous and not separately nearly continuous) nor jointly nearly continuous.

Example 4.5. Let f: $S \to R$ be defined as follows

$$f(x, y) = \begin{cases} 0, \text{ if } (-1 \leq x < 0 \wedge \;\wedge -1 \leq y \leq 1) \vee (x \in (0, 1\rangle \cap Q \wedge y \in \\ \qquad \in \langle -1, 1\rangle \cap Q) \\ 1, \text{ on the rest.} \end{cases}$$

This function is jointly (also separately) somewhat nearly continuous. However, f is neither jointly somewhat continuous nor separately somewhat continuous. Moreover, it is not jointly (nor separately) nearly continuous.

Now, we recall some results of other authors. From now up to the end of this paragraph, we assume that the considered space X is Baire, Y is second countable and Z is metric and f: $X \times Y \to Z$ is a function.

It is well known that

(4.6) Separate continuity does not imply joint continuity (even does not imply joint near continuity).

Martin [16], Theorem 1, p. 39, proved the following (see also [17], Theorem 1).

(4.7) If f is separately quasi-continuous, then f is jointly quasi-continuous.

He also showed that

(4.8) Joint quasi-continuity does not imply separate qasi-continuity.

In [16], Theorem 3, p. 41, (see also [26], Theorem 1) it is proved that

(4.9) If f is separately continuous, then f is quasi-continuous with respect to y.

Also in [16], Example 2, p. 42, it is shown that

(4.10) If f is symmetrically quasi-continuous (thus quasi-continuous with respect to y), then f need not be separately continuous.

Neubrunn showed in [19], Example 1, that

(4.11) Joint near continuity does not imply separate near continuity.

He also showed in [19], Example 2, that

(4.12) Separate near continuity does not imply joint near continuity.

In [17], Example, p. 99, it is shown that

(4.13) Separate somewhat continuity does not imply joint somewhat continuity.

Table 3

	jointly continuous functions	separately continuous functions	jointly quasi-continuous functions	separately quasi-continuous functions	functions quasi-continuous with respect to y	jointly somewhat continuous functions	separately somewhat continuous functions	jointly nearly continuous functions	separately nearly continuous functions	jointly somewhat nearly continuous functions	separately somewhat nearly continuous functions
jointly continuous functions	+	+ Def	+ Def	+ Def	+ Def	+ Def	+ Def	+ Def	+ Def	+ Def	+ Def
separately continuous functions	– (4.6	+	+ (4.7)	+ Def	+ (4.9)	+ (4.7)	+ Def	– (4.6)	+ Def	+ (4.7)	+ Def
jointly quasi-continuous functions	– (4.8)	– (4.8)	+	– (4.8)	– (4.1)	+ Def	– (4.8)	– (4.8)	– (4.2)	+ Def	– (4.3)
separately quasi-continuous functions	– (4.6)	– (4.1)	+ (4.7)	+	– (4.1)	+ (4.7)	+ Def	– (4.1)	– (4.1)	+ [4.7)	+ Def
functions quasi--continuous with respect to y	– (4.10)	– (4.10)	+ Def	– (4.8)	+	+ Def	– (4.3)	– (4.3)	– (4.3)	+ Def	– (4.3)
jointly somewhat continuous functions	– (4.8)	– (4.8)	– (4.2)	– (4.2)	– (4.2)	+	– (4.4)	– (4.4)	– (4.4)	+ Def	– (4.4)
separately somewhat continuous functions	– (4.6)	– (4.13)	– (4.13)	– (4.13)	– (4.2)	– (4.13)	+	– (4.2)	– (4.2)	?	– Def
jointly nearly continuous functions	– (4.11)	– (4.11)	– (4.11)	– (4.11)	– (4.11)	– (4.11)	– (4.11)	+	– (4.11)	+ Def	– (4.11)
separately nearly continuous functions	– (4.6)	– (4.12)	– (4.12)	– (4.12)	– (4.12)	– (4.12)	– (4.12)	– (4.12)	+	?	+ Def
jointly somewhat nearly continuous functions	– (4.8)	– (4.8)	– (4.2)	– (4.2)	– (4.2)	– (4.5)	– (4.5)	– (4.5)	– (4.5)	+	– (4.11)
separately somewhat nearly continuous functions	– (4.6)	– (4.8)	– (4.2)	– (4.2)	– (4.2)	– (4.13)	– (4.5)	– (4.5)	– (4.5)	?	+

We organize the results of § 4 in *Table 3*. The word "Def." means that the result easily follows from the definitions of the suitable functions. We put "?" if an answer is not known. Recall that the considered space X is Baire, Y is second countable and Z is metric, and $f: X \times Y \to Z$ is a function.

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SÚHRN

PREHĽAD VÝSLEDKOV TÝKAJÚCICH SA ZOVŠEOBECNENEJ SPOJITOSTI NA TOPOLOGICKÝCH PRIESTOROCH

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V práci je uvedený prehľad medzi nasledujúcimi typmi spojitosti: Kvázispojitosť, trochu-spojitosť, takmer-spojitosť, trochu-takmer-spojitosť.

РЕЗЮМЕ

ОБЗОР РЕЗУЛЬТАТОВ, КАСАЮЩИХСЯ ОБОБЩЕННОЙ НЕПРЕРЫВНОСТИ НА ТОПОЛОГИЧЕСКИХ ПРОСТРАНСТВАХ

З. Пийотровски, Йонгстовн

В работе приведен обзор связей между различными видами непреривностей, например: Квазинепрерывностью, почти непрерывностю, обобщенной непрерывностью и т.д.