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Dense Subsets of Product Spaces

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ABSTRACT. Two separability results are proved. It is shown *how* a dense, countable set is spread out in the product space. Thus, these results constitute a generalization of the classical Hewitt-Marczewski-Pondiczery theorem.

We shall start with the following two definitions. Assume $X = \Pi_{\alpha<\kappa}X_\alpha$ and let $D \subset X$.

We say that D is *thin* if whenever $x, y \in D$ and $x \neq y$, then $|\{\alpha < \kappa : x_\alpha \neq y_\alpha\}| > 1$.

A set D is said to be *very thin* if for every $\alpha < \kappa$ and $p \in X_\alpha$, we have $|\{x \in D : x_\alpha = p\}| \leq 1$.

Clearly, every very thin set is thin. In the product of two spaces, these two notions of thinness coincide.

Geometrically, in the case $\kappa = 3$, no two points of a thin set in I^3 can be on a *line* parallel to either of the axes; i.e., if D is thin and $x \in D \subset I^3, x = (x_0, y_0, z_0)$, then x is the only point in $[(\{x_0\} \times \{y_0\} \times Z) \cup (\{x_0\} \times Y \times \{z_0\}) \cup (X \times \{y_0\} \times \{z_0\})] \cap D$.

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Similarly, in the case of a very thin set D, no two points can lie on a *plane* parallel to one of three main planes, i.e., if $x \epsilon D \subset I^3$, then x is the only point in $[(\{x_0\} \times Y \times Z) \cup (X \times \{y_0\} \times Z) \cup (X \times Y \times \{z_0\})] \cap D$.

We shall now prove a separability theorem. For 2^ω-product of dense-in-themselves, separable spaces this Proposition shows how a dense, countable set may be spread out in the product, thus it generalizes the well-known Hewitt-Marczewski-Pondiczery theorem [1], p. 111.

PROPOSITION 1. Let X be the product of $\kappa (= 2^\omega)$ many separable spaces X_α with countable dense subsets $D_\alpha = \{x(\alpha, n) : n \epsilon \omega\}$, respectively. Then there is a countable, dense and thin subset of X.

PROOF. First, enumerate ${}^\omega 2$ as $\{\phi_\alpha : \alpha < \kappa\}$. For each $n \epsilon \omega$ and $p : {}^n 2 \to \omega$, let $y(n,p)$ be defined as follows: if $y = y(n,p)$ and $\alpha < \kappa$, then $y_\alpha = x(\alpha, p(\phi_\alpha \mid n))$. Let D be the set of all such $y(n,p)'s$. The set D is countable, dense and thin (in X).

Following J. C. Oxtoby [2] we say that a family B of non-empty open sets in a space is a *pseudo-base* or, shortly, a *π-base* if every non-empty open set contains at least one member of B. It is clear that every base is a π-base and every space having a countable π-base is separable.

The Stone-Čech compactification βN of naturals is an example of a space having a countable dense set of isolated points (hence, it has a countable π-base) but no countable base, cf. [2] p. 159.

The property of possessing a countable π-base is intermediate between that of having countable base and that of being a separable space. For metrizable spaces all three properties are equivalent, see [2].

Proposition 2. Let X be the product of 2^ω dense-in-themselves spaces X_α having countable π-bases. Then X contains a countable dense set D which is very thin.

Proof. Index a countable π-base $\mathcal{B}_\alpha = \{\bigcup(\alpha, m) : m < \omega\}$ and then recursively choose the dense set $D = \{y(n) : n < \omega\}$ so that if $y(n)_\alpha \in \bigcup(\alpha, m)$ then $y(n)_\alpha \neq y(k)_\alpha$ for all $k < n$.

Remark. In case of *metric* spaces X_α arguments go as follows: Begin with *any* countable dense subset $E = \{x(n) : n < \omega\}$ of X. Let $y(0) = x(0)$ and recursively, if $y(n)$ has been defined, choose $y(n+1)$ so that for each coordinate $\alpha < 2^\omega$ we have $\rho_\alpha(y(n+1)_\alpha, x(n+1)_\alpha) < \frac{1}{n}$ and $y(n+1)_\alpha \neq y(k)_\alpha$ for any $k \leq n$.

Question. Is the conclusion of Proposition 2 true if we assume that the spaces are only "separable" instead of "having countable π-bases"?

Apparently, Proposition 2 can be generalized to arbitrary products in fact, we have

Proposition 3. Let X be the product of 2^κ many dense-in-themselves spaces having π-weights κ. Then there is a dense, very thin set D of cardinality κ in X.

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References

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