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(GENERAL TOPOLOGY)

Full Blumberg Sets and Quasi-Continuity in Topological Spaces

by

Zbigniew PIOTROWSKI

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Summary. Neugebauer [7] proved that a function $f: I_0 \to R$ possesses a strong Blumberg set if and only if f is quasi-continuous on I_0, where R denotes the real line and $I_0 = \langle 0, 1 \rangle$. In this paper we give a characterization of quasi-continuous functions in a class of all functions from a Blumberg space for f (see Definition 3) into a regular space. So, it is a real generalization of Neugebauer's result.

We start from the following

DEFINITION 1 (compare [6] p. 186). A function f is called *quasi-continuous* (shortly: qc) if for each $x \in X$ and open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$, we have $A \cap \operatorname{Int} f^{-1}(H) \neq \emptyset$.

DEFINITION 2 (see [8] p. 157). A space X is called *a Baire space* if each non-void open set in X is of the second category.

DEFINITION 3. Let f be a function from a space X. We say that X is *a Blumberg space for f* if there exists a dense subset D of X such that the partial function $f|D$ is continuous. Such a set D is called a Blumberg set for f.

DEFINITION 4. A set D is called *a full Blumberg set for f* if D is a Blumberg set for f and, for every open set $A \subset X$, the set $f(D \cap A)$ is dense in $f(A)$ (compare the definition of a strong Blumberg set given in [7] p. 451).

At the beginning we compare the concept of a strong Blumberg set in the sense of Neugebauer [7] with the concept of a full Blumberg set in the sense mentioned above.

PROPOSITION. *Let $f: I_0 \to R$ and let D be a dense set in I_0. If for every interval $P \subset I_0$, the set $f(D \cap P)$ is dense in $f(P)$, then for every open set $G \subset I_0$, the set $f(D \cap G)$ is dense in $f(G)$.*

In fact, each open set in I_0 is the union of open intervals, and thus the proposition follows.

This means that each strong Blumberg set in the sense of Neugebauer is a full Blumberg set. Therefore our Definition 4 is a natural extension of Neugebauer's one. The converse implication does not hold (the author wishes to thank the referee for the example below). Indeed, define $f: \langle 0, 1\rangle \to R$ as follows: $f(x)=x$ for $x \in \langle 0, \frac{1}{2}\rangle$ and $f(x)=x+1$ for $x \in (\frac{1}{2}, 1\rangle$ and let $D=\langle 0, 1\rangle \setminus \{\frac{1}{2}\}$. In fact, taking $\langle \frac{1}{2}, 1\rangle$ as the interval we have that $\frac{1}{2} \notin f(D \cap \langle \frac{1}{2}, 1\rangle)$.

THEOREM 1. *If $f: X \to Y$ is* qc, *D is a dense subset of X and A is an open subset of X, then $f(D \cap A)$ is dense in $f(A)$.*

Proof. Let A be an open set in X containing x_0 and let H be an open set in Y containing $y_0 = f(x_0)$. Since f is qc, $A \cap \operatorname{Int} f^{-1}(H) \neq \emptyset$. Put $Q = A \cap \operatorname{Int} f^{-1}(H)$. The set Q is open. It is clear that $f(Q) \subset H$. Now, $H \cap f(D \cap A) \supset H \cap f(D \cap Q) = f(D \cap Q)$. By the density of D in X, the last set is non-void, whence $H \cap f(D \cap A) \neq \emptyset$ which means that $f(D \cap A)$ is dense in $f(A)$, by the choice of H (see [4], Theorem 5, p. 38).

COROLLARY 1. *If D is a Blumberg set for the* qc *function $f: X \to Y$, then D is a full Blumberg set for f.*

THEOREM 2. *Let f be a function from a space X which is a Blumberg space for f into a regular space Y. If f possesses a full Blumberg set D, then f is* qc *on X.*

Proof. Let x_0 belong to an open subset A of X and H be an open subset of Y containing $y_0 = f(x_0)$. Since $f(A \cap D)$ is dense in $f(A)$ and $y_0 \in f(A)$ there exists such a point $x_1 \in A \cap D$ that $f(x_1) = y_1$ belongs to H. Let G be an open subset of Y such that $y_1 \in G \subset \bar{G} \subset H$ (such a set G exists by the regularity of Y). Now, $f|D$ is continuous, D being a Blumberg set for f. Therefore there exists an open (in D) set $B \subset D$, containing x_1 such that $f(B) \subset G$. Then $B \cap A$ is an open (in D) neighbourhood of x_1, for which $f(B \cap A) \subset G$. Let B' be an open subset of X such that $B' \cap D = B \cap A$ and $B' \subset A$. We have $f(B' \cap D)$ is dense in $f(B')$ since B' is an open subset of X. Hence we get $f(B') = \overline{f(B' \cap D)} = \overline{f(B \cap A)} \subset \bar{G} \subset H$. Accordingly $B' \subset f^{-1}(f(B)) \subset f^{-1}(H)$ hence $B' \subset \operatorname{Int} f^{-1}(H)$, B' being open in X. But $B' \subset A$, so finally $B' \subset A \cap \operatorname{Int} f^{-1}(H)$. Since $x_1 \in A \cap D$ and $x_1 \in B$, we have $x_1 \in B'$, by the definition of B'. Thus $B' \neq \emptyset$ which finishes the proof.

The main result of this paper is the following

THEOREM 3. *Let f be a function from a space X which is a Blumberg space for f into a regular space Y. The function f possesses a full Blumberg set if and only if f is* qc.

Proof. It follows from Corollary 1 and Theorem 2. One can obtain some corollaries taking the main result of [10] and the theorem of [3] p. 667.

Remark 1. The assumption of quasi-continuity of f cannot be weakened. Recall that a function $f: X \to Y$ from X into Y is called somewhat continuous ([5] p. 6) if for each set $V \subset Y$, open in Y, such that $f^{-1}(V) \neq \emptyset$ there exists an open set $U \subset X$, $U \neq \emptyset$ with $U \subset f^{-1}(V)$. There is a counterexample that using a somewhat continuous function instead of quasi-continuous one Theorem 1 as well as Corollary 1 become false. Indeed, define $f: I_0 \to I_0$ by $f(x)=0$ if $x \in \langle 0, \frac{1}{4}\rangle$ or $x \in (\frac{1}{2}, 1\rangle$ and x is rational; $f(x)=1$ if $x \in (\frac{1}{4}, \frac{1}{2}\rangle$ or $x \in (\frac{1}{2}, 1\rangle$ and x is irrational. Such a function f is somewhat continuous (but not qc) and Theorem 1 (as well as Corollary 1) does not hold for f.

Remark 2. Some similar problems which concern Blumberg sets (but not in connection with quasi-continuous functions) were discussed and investigated by Szymański in [9]. Also Alas has announced [1] a result being a generalization of Proposition 1.7 of [10].

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INSTITUTE OF MATHEMATICS, UNIVERSITY, PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW
(INSTYTUT MATEMATYCZNY, UNIWERSYTET WROCŁAWSKI)

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З. Пиотровски, **Сильное множество и квази-непрерывность в топологических пространствах**

Содержание. Нейгебауер [7] доказал' что для того чтобы для функции $f: I_0 \to R$ существовало сильное множество Бламберга, необходимо и достаточно чтобы она была квази-непрерывной на I_0 где R обозначает действительную прямую а $I_0 = \langle 0, 1\rangle$. В этой работе дается характеризация квази-непрерывных функций в классе функции из пространства Бламберга для функции (смотри Дефиницию 3) в регулярное пространство. Поэтому наша теорема является обобщением результата Нейгебауера.