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ON KEMPISTY'S GENERALIZED CONTINUITY

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We study an analogue of symmetric quasicontinuity in the case of a function defined on the Cartesian product of three spaces. Some results of S. Kempisty, N. F. G. Martin and the second named author are generalized.

1. Introduction.

S. Kempisty [2] introduced the notion of symmetric quasi-continuity extending classical results of R. Baire and H. Hahn on separate versus joint continuity.

Kempisty's ideas have been developed, subsequently by N. F. G. Martin [4], T. Neubrunn [5], and the second-named author [8].

This note is a continuation of such studies. We shall present an analogue of symmetric quasi-continuity in the case of a function f defined on 3-space, or more generally, on the Cartesian product of three (topological) spaces.

This way we study a new continuity-like property. The considered class of functions is logically intermediate between continuous and quasi-continuous [4] functions. This new concept concerns the functions defined on the product of three spaces only.

Let us recall that a function $f: X \to Y$ is said to be *quasi-continuous* ([4], [2]) if for every $x \in X$ and for every neighborhood U of x and for every neighborhood V of $f(x)$, we have: $U \cap \operatorname{Int} f^{-1}(V) \neq \emptyset$.

Following N. F. G. Martin ([4]), [2]) we say that a function $f: X \times Y \to Z$ (X, Y and Z are arbitrary spaces) is *quasi-continuous with respect to* x if for every (x, y) in $X \times Y$, for every neighborhood $U \times V$ of (x, y) and for every such that if U' is a neighborhood of x_0 with $U' \subset U$ and V' is an open, nonempty set such that $V' \subset V$, then we have

$$g(U' \times V') \cap (T \setminus N) \neq \emptyset \tag{*}$$

Let $\{U_n\}$ be a local countable basis at x_0, say, contained in U.

Let $N_0 \subset T$ be an open set, such that

$$g(x_0, s_0) \in N_0 \subset \overline{N_0} \subset N.$$

By quasi-continuity of g_{x_0} at s_0 we have

$$W_0 = \operatorname{Int} g_{x_0}^{-1}(N_0) \cap V \neq \emptyset .$$

Let $F \subset S$ be the first category set mentioned in the assumptions of the Lemma i.e., g_s is not continuous for $s \in F$.

$$\text{Now, let } W = (\operatorname{Int} g_{0_x}^{-1}(N_0) \cap V) \setminus F$$

Put
$$K = \{s \in W : g_s \text{ is continuous}\}$$

It is easy to see that $K = W$. In fact, by the definition of K, $K \subset W$. The converse inclusion easily follows from the definition of W—elements of $F \cap W_0$ (for which g_s is not continuous) do not belong to W.

The set K, as the complement of the first category set F in the open nonempty set W_0, is residual. Hence K is a second category set, S being Baire.

Now we define the sets A_n as follows:

$$A_n = \{s : s \in K, U_n \subset g_s^{-1}(N_0)\}.$$

We shall show that $K = \bigcup_{n=1}^{\infty} A_n$. The inclusion $\bigcup_{n=1}^{\infty} A_n \subset K$ is obvious. If $s \in K$, then $s \in W$, hence $g_{x_0}(s) \in N_0$. So $g_s(x_0) \in N_0$ and $x_0 \in g_s^{-1}(N_0) \cap U$, but $g_s^{-1}(N_0) = \operatorname{Int} g_s^{-1}(N_0)$, by continuity of g_s. Thus, there is an index n such that $U_n \subset g_s^{-1}(N_0)$.

Now, we shall show that for every n, the set A_n is nowhere dense in W.

Let $G \subset V$ be an arbitrary non-empty open set. Let us form $U_n \times G$ for any given n.

such that if U' is a neighborhood of x_0 with $U' \subset U$ and V' is an open, nonempty set such that $V' \subset V$, then we have

$$(*) \qquad g(U' \times V') \cap (T \setminus N) \neq \emptyset$$

Let $\{U_n\}$ be a local countable basis at x_0, say, contained in U.

Let $N_0 \subset T$ be an open set, such that

$$g(x_0, s_0) \in N_0 \subset \overline{N_0} \subset N.$$

By quasi-continuity of g_{x_0} at s_0 we have

$$W_0 = \operatorname{Int} g_{x_0}^{-1}(N_0) \cap V \neq \emptyset.$$

Let $F \subset S$ be the first category set mentioned in the assumptions of the Lemma i.e., g_s is not continuous for $s \varepsilon F$.

$$\text{Now, let } W = (\operatorname{Int} g_{0_x}^{-1}(N_0) \cap V) \setminus F$$

Put $$K = \{s \varepsilon W : g_s \text{ is continuous}\}$$

It is easy to see that $K = W$. In fact, by the definition of K, $K \subset W$. The converse inclusion easily follows from the definition of W—elements of $F \cap W_0$ (for which g_s is not continuous) do not belong to W.

The set K, as the complement of the first category set F in the open nonempty set W_0, is residual. Hence K is a second category set, S being Baire.

Now we define the sets A_n as follows:

$$A_n = \{s : s \in K, U_n \subset g_s^{-1}(N_0)\}.$$

We shall show that $K = \bigcup_{n=1}^{\infty} A_n$. The inclusion $\bigcup_{n=1}^{\infty} A_n \subset K$ is obvious. If $s \in K$, then $s \in W$, hence $g_{x_0}(s) \in N_0$. So $g_s(x_0) \in N_0$ and $x_0 \in g_s^{-1}(N_0) \cap U$, but $g_s^{-1}(N_0) = \operatorname{Int} g_s^{-1}(N_0)$, by continuity of g_s. Thus, there is an index n such that $U_n \subset g_s^{-1}(N_0)$.

Now, we shall show that for every n, the set A_n is nowhere dense in W.

Let $G \subset V$ be an arbitrary non-empty open set. Let us form $U_n \times G$ for any given n.

It follows from (*) that there is $(x_1, s_1) \in U_n \times G$ such that $g(x_1, s_1) \notin N$. We choose a neighborhood N_1 of $g(x_1, s_1)$ such that $N_0 \cap N_1 = \varnothing$.

By quasi-continuity of g_{x_1} at s_1, there is a nonempty open set $G_1 \subset G$ such that for any $s \in G_1$, $g(x_1, s) \in N_1$ and hence $g(x_1, s) \notin N_0$.

Thus for any $s \in G_1, x_1 \notin g_s^{-1}(N_0)$. This implies $U_n \not\subset g_s^{-1}(N_0)$ so that $s \notin A_n$. So, $G_1 \cap A_n = \varnothing$. This shows that A_n is nowhere dense.

Finally, we obtain that the set $K = \bigcup_{n=1}^{\infty} A_n$ is of first category (!). This is a contradiction to our assumption that K is a second category subset of the Baire space S, and the proof of Lemma 3 in finished.

For the sake of completeness we shall prove the following.

LEMMA 3. *Let X, Y, Z and T be spaces and let* $f: X \times Y \times Z \to T$ *be a function. Then f is x-continuous if and only if* $g: X \times S \to T$ *is quasi-continuous with respect to x, where* $S = Y \times Z$ *and* $g(x, (y, z)) = f(x, y, z)$.

Proof. Suppose f is x-continuous at (p_0, q_0, r_0). Then for every neghborhood N_0 of $f(p_0, q_0, r_0)$ and for every neighorhood $U_0 \times V_0 \times W_0$ of (p_0, q_0, r_0) there is a neighborhood U_0' of p_0 with $U_0' \subset U_0$ and nonempty open sets V_0' and W_0' with $V_0' \subset V_0$ and $W_0' \subset W_0$ such that for all $(x, y, z) \in U_0' \times V_0' \times W_0'$ we have $f(x, y, z) \in N_0$.

But $V_0' \times W_0'$ is then an open and nonempty subset in $V_0 \times W_0$ (in the space S). Hence g is quasi-continuous with respect to x *at* (p_0, s_0), where $s_0 = (q_0, r_0)$.

Conversely, if g is quasi-continuous with respect to x at (p_0, s_0), then for every neighborhood N_0 of $g(p_0, s_0)$ and for every neighborhood $U_0 \times G_0$ of (p_0, s_0) there is a neighborhood U_0' of p_0, with $U_0' \subset U_0$ and a nonempty open set $V_0' \subset G_0$ so that $g(U_0' \times V_0') \subset N_0$. Recall that $S = Y \times Z$; hence the respective projection maps onto Y and Z send s_0 onto q_0 and r_0 respectively. Since the projections Pr_Y and Pr_Z preserve openess of U_0' and V_0' respectively, it follows that $Pr_Y G_0$ and $Pr_Z G_0$ are neighborhood q_0 and r_0, respectively. This proves that f is x-continuous at (p_0, q_0, r_0)

THEOREM 1. *Lef X be first countable, Y be Baire, Z be second countable in a neighborhood of any of its points and such that* $Y \times Z$ *is Baire and let T be regular.*

If $f: X\times Y\times Z\rightarrow T$ is a function such that:

(1) f is continuous at points of $X\times\{y\}\times\{z\}$, for $y\in Y$ and $z\in Z$, except, possibly, for a first category set of points (y, z) in $Y\times Z$, and

(2) f is quasi-continuous at points of $\{x\}\times Y\times\{z\}$ for all $x\in X$ and all $z\in Z$, and

(3) f is quasi-continuous at points of $\{x\}\times\{y\}\times Z$ for $x\in X$ except, possibly, for a first category set of points (x, y) in $X\times Y$,
then f is x-continuous.

Proof. By (2) and (3), it follows from Lemma 1 that f_x is quasi-continuous at every (y, z) in $Y\times Z$. Now, by (1), it follows from Lemma 2 that the function g, that is defined by $g(x, (y, z))=f(x, y, z)$ is quasi-continuous with respect to x. Hence, we conclude, by Lemma 3, that f is x-continuous.

The above Theorem has an interesting and clear geometrical interpretation. Namely we have the following:

COROLLARY 1. *Let $f: R^3\rightarrow R$ be a function satisfying the following conditions*:

(1) There is a residual set of lines parallel to the x-axis consisting of points at which f is continuous, relative to the line.

(2) For any line parallel to the y-axis f is quasi-continuous relative to the line.

(3) There is a residual set of lines parallel to the z-axis consisting of points at which f is quasi-continuous, relative to the line.

Then f is x-continuous.

Remark 1. Since projections, or more generally, open and continuous images of Baire spaces are Baire, both the factors X and Y must be Baire in order for $X\times Y$ to be so. Further, the Cartesian product of Baire, even metric (hence first countable) spaces, does not have to be Baire. Theorem 5.1 of [1] gives eight properties for X and Y so that $X\times Y$ is a Baire space. See also [7], Proposition 9, where another condition, Somewhat similar to Theorem 5.1 (viii) of [1] is given.

The next Corollary follows from the above results and Theorem 6 of [6].

COROLLARY 2. *Let X be first countable, $Y_1, \ldots, Y_n$ be pseudo-complete spaces which are second countable in a neighborhood of any of their points and let T be regular.*

If $f : X \times \prod_{i=1}^{n} Y_n \to T$ is a function such that

(1) f is continuous at points of $X \times \{y_1\} \times \{y_2\} \times \ldots \times \{y_n\}$ for $y_i \in Y_i$, respectively except possibly for a first category set of points $(y_1, y_2, \ldots, y_n)$ in $Y_1 \times Y_2 \times \ldots \times Y_n$ and

(2) f is quasi-continuous at points of $\{x\} \times Y_1 \times \{y_2\} \times \ldots \{y_n\}$, $\{x\} \times \{y_1\} \times \times Y_2 \times \{y_3\} \times \ldots \times \{y_n\}, \ldots, \{x\} \times \{y_1\} \times \{y_2\} \times \ldots \times Y_n$, for $x \in X$, $y_i \in Y_i$, $i = 1, \ldots n$,

then for every neighborhood $U \times \prod_{i=1}^{n} V_i$ of every $(x^1, y^1, y^2, \ldots, y^n) \in X \times \prod_{i=1}^{n} Y_n$ and for every neighborhood N *of* $f(x^1, y^1, \ldots, y^n)$ there is a neighborhood U' of x', with $U' \subset U$ and nonempty open sets V_i' $i = 1, \ldots, n$, with $V_i' \subset V_i$ such that $f(U' \times \prod_{i=1}^{n} V_i) \subset N$.

It appears that under a somewhat stronger assumption on T than one given in Theorem 1, namely that T is metric, while the conditions on X, Y, Z and f remain the same we have the following result that gives a description of the set of points of continuity in hyperspaces of $X \times Y \times Z$ resulting by fixing $x's$ from X.

Namely, let us notice that the function f is then quasi-continuos (compare [5]). Now by [3] f has a residual set of points of continuity. So if we additionally assume that the domain $X \times Y \times Z$, is Baire, see Remark 1, then $C(f)$ (=the set of points of continuity of f) is dense G_δ set in $X \times Y \times Z$.

However, it is possible to get yet a better description of $C(f)$. In fact, let us fix an arbitrary $x \in X$. Now consider the oscillation function $o : (y, z) \to \omega(x, y, z)$. The function o is lower semi-continuous; the open set $\{(y, z) : \omega(x, y, z) < \frac{1}{n}\}$ is dense in $Y \times Z$ and in fact, standard arguments show that this set is of G_δ class. Concluding, we have

THEOREM 2. *Let T be a metric space, and let X, Y, Z and f satisfy all the conditions given in Theorem 1. Then for every $x \in X$, the set of points of continuity of is a dense G_δ subset in $\{x\} \times Y \times Z$.*

COROLLARY 3. *Let $f: R^3 \rightarrow$ satisfy all the assumptions given in Corollary* 1. *Then there is a dense G_δ set of points of continuity in every plane parallel to the* yz-plane.

In a sense, Theorem 2 gives the best description of how the continuity points are spread out in subspaces of $X \times Y \times Z$. That is, there are functions $f: R^3 \rightarrow R$, that are separately continuous, hence satisfy all the assumptions on f in Corollary 3 and yet discontinuous at every point of line parallel to one of the main axes. In the following Example which is a part of the folklore we construct such a function.

EXAMPLE. *Let $g: (x, y) \rightarrow R$ be any separately continuous function which is discontinuous at* (0.0). *Then $f: (x, y, z) \rightarrow R$, defined by $f(x, y, z) = g(x, y)$ is discontinuous at every point of the z-axis.*

Remark 2. Quasi-continuity has been introduced by G. Volterra see R. Baire, Ann. Mat. Pura Appl. 3 (1899) p. 75, while studying separate versus joint continuity problems. Somewhat similar class of functions, namely *B*-functions have been studied by H. Hahn, « Reelle Funktionen » Leipzig, 1932 pp. 325-338, see also MR 84j #54009 (by G. Mägerl).

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