Functions Which Preserve Lebesgue Spaces*

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Abstract

In this note we show that functions which preserve the property of being a Lebesgue space are precisely the class of metric preserving functions.

1 Introduction and Definitions

The idea of preserving a metric is included in almost every introduction to metric spaces. It is typical to have an exercise to prove that if ρ is a metric on M, then $\rho 1+\rho$ is bounded and a metric on M. In this exercise ρ is composed with the function $f(x) = x1 + x$. As we can see, the definition of metric preserving functions is a generalization of this.

Let $\mathbb{R}^+$ denote the non-negative real numbers. A function

$$f : \mathbb{R}^+ \to \mathbb{R}^+$$

is called metric preserving if for every metric space (M, ρ) the function defined by $f \circ \rho$ is also a metric on M.

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One description of metric preserving functions is based on the idea of triangle triplets [**?**].

For three non-negative numbers a, b, and c we say (a, b, c) is a triangle triplet if $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$. A function f is metric preserving if and only if $f^{-1}(0) = \{0\}$ and $(f(a), f(b), f(c))$ is a triangle triplet whenever (a, b, c) is one.

A consequence of this definition is the following lemma. It gives a lower bound of sorts for the image of y larger than some x. This result is something we will use later.

Let f be a metric preserving function and $x \in \mathbb{R}^+$. If $y > x$, then $f(y) > f(x)/2$.

Proof Since $y > x$, then (x, y, y) is a triangle triplet. Because f preserves metrics $(f(x), f(y), f(y))$ is also a triangle triplet. This means

$$f(x) \leq f(y) + f(y)$$

or $f(y) \geq f(x)/2$.

The literature on metric preserving functions goes at least as far back as 1935 [**?**]. The study of these functions have picked up in recent years. These includes several examples of pathological functions ([**?**] and [**?**]) and applications ([**?**] and [**?**]). For reference, see the surveys by Doboš [**?**] or Vallin [**?**]. Our goal is this paper is to show another application by tying the subject of metric preserving functions with the concept of a Lebesgue space. The definition of a Lebesgue space (from [**?**]) follows:

Let (X, d) be a metric space. A Lebesgue number for an open cover $\mathcal{U}$ of X is an $\varepsilon > 0$ such that for each point $p \in X$, the open ball $B_d(p, \varepsilon) = \{x \in X : d(p, x) < \varepsilon\}$ is contained in at least one member of $\mathcal{U}$.

A Lebesgue space (L-space) is a metric space such that every open cover of the space has a Lebesgue number.

The fact that every compact metric space is an L-space means there are an overabundance of "easy" examples of Lebesgue spaces. For a non-compact example, let $X = \mathbb{N}$ with the discrete metric. For any open cover of X, if we let $\varepsilon = 1/2$ we're done.

In what follows we will make use of the following theorem from Levine [**?**].

Let (X, d) be a metric space. On the collection of closed subsets of X let $D(E, F) = \inf\{d(x, y) : x \in E and y \in F\}$. Given the following statements

1. (X,d) is compact

2. for every E, F non-empty, disjoint, and closed there exists $x \in E$ and $y \in F$ such that $D(E,F) = d(x,y)$

3. $D(E,F) > 0$ for all non-empty, disjoint, closed sets

4. (X,d) is an L-space (the definition in [**?**]: for an open cover Φ of X, there exists an $\eta > 0$ such that for A a subset of X with $diam(A) < \eta$, then $A \subset O$ for some $O \in \Phi$.)

5. every function $f : (X,d) \to (X^*, d^*)$ (where (X^*, d^*) is arbitrary) is uniformly continuous

6. (X,d) is complete

we have $1 \to 2 \to 3 \leftrightarrow 4 \leftrightarrow 5 \to 6$.
Specifically, we will use $3 \leftrightarrow 4$.

2 Results

We begins this section with our motivation for this paper. The following theorem comes from [**?**].

Let (X,d) be a metric space, $d^* = d1 + d$, and $d^{**} = \min\{d, 1\}$. Then (X,d) is an L-space if and only if (X,d^*) is an L-space if and only if (X,d^{**}) is an L-space.

It is obvious that both $f(x) = x1 + x$ and $g(x) = \min\{1, x\}$ are metric preserving functions. This brings us to the following questions: What functions on d will preserve the L-space property? Will all metric preserving functions preserve the L-space property? Are there functions which preserve the L-space property but are not metric preserving? These will be answered in the next theorem.

Let (X,d) be an L-space. The function $f : \mathbb{R}^+ \to \mathbb{R}^+$ has the property that $(X, f \circ d)$ is an L-space if and only if f is a metric preserving function.

Proof Let (X,d) be an L-space and f a metric preserving function. Take two non-empty, disjoint, closed sets E and F in $(X, f \circ d)$. If f is continuous, then E and F are also closed in (X,d). Since (X,d) is an L-space, we have $D(E,F) > 0$. Thus $\alpha = \inf\{d(x,y) : x \in Eandy \in F\} >$

0. Now then any $d(x,y) \geq \alpha$ (where $x \in E$ and $y \in F$) and so $f(d(x,y)) > f(\alpha)/2$ by Lemma 3. Hence

$$D_f(E,F) = \inf\{f(d(x,y)) : x \in Eandy \in F\}$$

is positive. If f is not continuous, then by Proposition 3.2 in [**?**], $(X, f \circ d)$ is a uniformly discrete metric space, and such spaces are L-spaces. Thus $(X, f \circ d)$ is an L-space.

The other direction is obvious.

Since functions which preserve metrics are equivalent to functions which preserve L-spaces there are already in place several interesting results. To begin with, we have several descriptions of these types of functions. Of course, we have already given one description in the form of triangle triplets. In addition, if $f(0) = 0$ and f is concave, then f is metric preserving. Lastly, if $f(0) = 0$ and there is a positive a such that $a \leq f(x) \leq 2a$ for $x > 0$ then f is metric preserving.

There are also pathological examples in place. Doboš and Piotrowski in [**?**] constructed a continuous, nowhere differentiable metric preserving function. Vallin in [**?**] showed that for any given measure zero, $\mathcal{G}_\delta$ set G in $\mathbb{R}^+$ there exists a continuous, everywhere differentiable (in the extended sense) f which is metric preserving and has $|f'(x)| = \infty$ precisely on $\{0\} \cup G$.

The next question is about the image of the Lebesgue number under a metric preserving f. Given a space (X,d), an open cover $\mathcal{U}$, a Lebesgue number $\varepsilon > 0$, and a metric preserving function f, will $\mathcal{U}$ have Lebesgue number $f(\varepsilon)$ under the metric $f \circ d$? The answer is yes, if f is increasing, but in general, no.

Let $X = \mathbb{R}$ with d the Euclidean metric and

$$\mathcal{U} = \{(n - 3/2, n + 3/2 : n \in \mathbb{Z})\}.$$

Note: ε can be any number between 0 and 1. We define our metric preserving function as follows

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x < 3/2 \\ 1/2 & x \geq 3/2 \end{cases}.$$

So $f(\varepsilon) = \varepsilon$, if $0 < \varepsilon \leq 1$, but $B_{f \circ d}(n, 2/3)$ will not be contained in any member of $\mathcal{U}$ for any n. In this instance we can have $\varepsilon = 1$ in the Euclidean metric but using $f \circ d$ the Lebesgue number must be smaller than or equal to $1/2$.

Although we may not be able to use $f(\varepsilon)$ for the Lebesgue number for $f \circ d$ we can at least provide some idea for the new value.

If (X, d) has open cover $\mathcal{U}$ with Lebesgue number ε and f is a metric preserving function, then for $(X, f \circ d)$, any value between 0 and $f(\varepsilon)/2$ can be used as a Lebesgue number for $\mathcal{U}$.

Proof This is a consequence of the Lemma 3. Recall the lemma gives $f(x)/2 < f(y)$ if $x < y$. So for some f we can have $B_{f \circ d}(x, f(\varepsilon)/2)$ containing points that are not in any of sets in U. However, if $\eta < f(\varepsilon)/2$ then for $y > \varepsilon$ we have $f(y) > \eta$ so

$$B_{f \circ d}(x, \eta) \subset B_d(x, \varepsilon)$$

for any point x.

Questioning in the opposite direction, if $f(\varepsilon) = \widehat{\varepsilon}$ and $\widehat{\varepsilon}$ is a Lebesgue number for $\mathcal{U}$ under $f \circ d$, is ε a Lebesgue number for $\mathcal{U}$ under d? No, the previous example shows this since $f(2) = 1/2$. We can, however, say that if $\widehat{\varepsilon}$ is a Lebesgue number for $\mathcal{U}$ under $f \circ d$, then $\varepsilon_0 = \inf \{f^{-1}(\widehat{\varepsilon})\}$ will be one under d.

In closing, we note that the previous result will not necessarily hold if, instead of considering all metric spaces, we narrow our focus to the real line with the Euclidean metric (a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ preserves $(\mathbb{R}, Eucl.)$ if and only if $f^{-1}(0) = \{0\}$ and $(f(a), f(b), f(a+b))$ is a triangle triplet for all positive a and b). Two recent papers ([**?**] and [**?**]) have dealt with metric preservation on just the real line with the Euclidean metric and both have examples of Euclidean preserving functions where

$$\liminf_{x \to \infty} f(x) = 0$$

So take for example the closed interval $[0, 1]$ and let

$$\mathcal{U} = \{[0, 1/2), (1/3, 2/3), (1/2, 1]\}.$$

For this $\mathcal{U}$ let $\varepsilon > 0$ be our Lebesgue number. By rescaling if necessary, we can use any of the functions from [**?**] or [**?**] to find an open interval V of radius $\eta > \varepsilon$ with V not a subset of any member of $\mathcal{U}$, yet

$$f(\eta) \leq f(\varepsilon).$$

In fact, we can say the following.

Let V, a subset of $(\mathbb{R}, |\cdot|)$ be a Lebesgue space. Let $\mathcal{U}$ be an open cover of V with Lebesgue number ε. Then the only way for V to be a Lebesgue space with respect to $\mathcal{U}$ in the metric $f(|\cdot|)$ for *all* f which preserve $(\mathbb{R}, |\cdot|)$ with $\liminf_{x\to\infty} f(x) = 0$ is for V to contain all open intervals of radius ε or more, or contain $\mathbb{R}$.

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