Mibu-type theorems.

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ABSTRACT. A Mibu-type theorem is any result in which a class of not necessarily continuous y-sections \( f_y \) is specified, so that for any \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) having all x-sections \( f_x \) continuous, the set \( C(f) \) of point of continuous, the set \( C(f) \) of point of continuity of \( f \) is a dense, \( G_d \) subset of \([0, 1] \times [0, 1]\).

In this paper a different proof of a Mibu-type theorem is provided. For the reader's convenience though, we quote the most needed and relevant definitions and theorems.

We illustrate the presentation of the results with a few examples showing that some of the theorems are best possible. Two questions are posed.

§1. INTRODUCTION

Let us start from the following:

(*) Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be a function having all x-sections \( f_x \) continuous. What must all y-sections \( f_y \) be in order for \( f \) to have a dense \( G_d \) set \( C(f) \) of points of continuity?

Of course, if all y-sections are continuous, then by a theorem of R. Baire [Ba] such an \( f \) is of first class of Baire, and as such it has a dense \( G_d \) set \( C(f) \), see [P2] for further generalizations.

But what if we weaken, a bit, the assumptions pertaining to the sections?

As an example, [P1] shows one cannot relax these assumptions too much, since there is a function \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) having: all but countably many x-sections (resp. y-sections) continuous, whereas these countably many x-sections (resp. y-sections) have only finitely many points of discontinuity (thus all x-sections and all y-sections are being of first class) while the set \( C(f) = \phi \).
Now, assume \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) has all of its \( x \)-sections \( f_x \) continuous and all of its \( y \)-sections \( f_y \) of first class of Baire. What is the set \( C(f) \)?

There is no general theory that would provide an "automatic" answer. The celebrated theorem of Baire, later generalized by H. Lebesgue, K. Kuratowski and D. Montgomery asserts that such an \( f \) is of second class of Baire. But second class of Baire functions may have empty set of points of continuity(!); take the "salt & pepper" function, \( f(x) = 0 \), if \( x \) is rational \( f(x) = 1 \), if \( x \) is irrational.

Before we answer the above specific question as well as provide a few answers to a general problem (*) mentioned at the beginning of the Introduction, we need a couple of definitions.

§2. Definitions.

A space \( X \) is called Baire if every open nonempty subset of \( X \) is of second category.

If \( A \subset X \) and \( U \) is a collection of subsets of \( X \), then \( st(A, U) = \bigcup \{U \in U : U \cap A \neq \emptyset\} \).

For \( z \in X \), we write \( st(z, U) \) instead of \( st(\{z\}, U) \).

A sequence \( \{G_n\} \) of open covers of \( X \) is called a development of \( X \) if for each \( z \in X \) the set \( \{st(z, G_n) : n \in N\} \) is a base at \( z \).

A developable space is a space which has a development\(^1\)

A Moore space is a regular developable space.

A function \( f : X \to Y \) is called quasi-continuous at a point \( x \in X \) if for each open sets \( A \subset X \) and \( B \subset f(X) \), where \( x \in A \) and \( f(x) \in B \) we have \( A \cap \text{Int} f^{-1}(B) \neq \emptyset \).

A function \( f : X \to Y \) is called quasi-continuous if it is quasi-continuous at each point of \( X \).\(^2\)

Given spaces \( X, Y \) and \( Z \), a function \( f : X \times Y \to Z \) is said to be quasi-continuous at \((p, q) \in X \times Y \) with respect to the variable \( y \) if for every neighborhood \( N \) of \( f(p, q) \) and for every neighborhood \( U \times V \) of \((p, q) \) there exists a neighborhood \( V^1 \) of \( q \) with \( V^1 \subset V \) and a nonempty open \( U^1 \subset U \) such that for all \((x, y) \in U^1 \times V^1 \) we have \( f(x, y) \in N \). If \( f \) is quasi-continuous with respect to the variable \( y \) at every point of its domain, we say that \( f \) is quasi-continuous with respect to \( y \).

Given a metric space \( M \), a function \( f : X \to M \) is called of first class (in the sense of G. Debs) if for every \( \varepsilon > 0 \), for every nonempty subset \( A \subset X \), there is a nonempty set \( U \), open in \( A \), such that \( \text{diam}(f(U)) \leq \varepsilon \).

For "nice" spaces, say \( X = Y = \mathbb{R} \), being the domain and the range of \( f \), respectively we have the following diagram, where "\( \to \)" denotes the inclusion:

\(^1\)Every metric space is developable; take the family \( B_\varepsilon \) of balls of diameter less than \( \varepsilon \) as a development.

\(^2\)See [Ba] p. 95, see also [HT] and [Tr].
where PWD stands for the pointwise discontinuity; a function $f$ is pointwise discontinuous if $C(f)$ is dense.

§3. SOME MIBU-TYPE THEOREMS.

We are now ready to give some answers to the spectacular question (*) mentioned at the beginning of the Introduction.

Due to the fact, that Y. Mibu [Mi] was the first to answer this question, for $f_y$'s not being necessarily continuous, any positive answer to (*) we shall call a Mibu-type theorem.

Consider the following four theorems

**Theorem 1.** [Mi] – Mibu’s First Theorem.

Let $X$ be first countable, $Y$ be Baire and such that $X \times Y$ be Baire. Given a metric space $M$. If $f : X \times Y \to M$ is separately continuous, then $C(f)$ is a dense $G_\delta$ subset of $X \times Y$.

**Theorem 2.** [Mi] – Mibu’s Second Theorem.

Let $X$ be second countable, $Y$ be Baire and such that $X \times Y$ is Baire. Given a metric space $M$. If $f : X \times Y \to M$ has:

a) all $x$-sections $f_x$ PWD and;

b) all $y$-sections $f_y$ continuous; then $C(f)$ is a dense $G_\delta$ subset of $X \times Y$.

**Theorem 3.** [De] – Deb’s Theorem.

Let $X$ be first countable, $Y$ be a special $\alpha$-favorable space (thus Baire), $X \times Y$ be Baire. Given a metric space $M$.

If $f : X \times Y \to M$ has:

a) all $x$-sections $f_x$ of first class (in the sense of G. Debs) and;

b) all $y$-sections $f_y$ continuous; then $C(f)$ is a dense $G_\delta$ subset of $X \times Y$.

**Theorem 4.** [P1], Theorem B.

Let $X$ be first countable, $Y$ be Baire and $Z$ be Moore. If $f : X \times Y \to Z$ has:

a) all $x$-sections $f_x$ quasi-continuous and;

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3Theorem 4 implies Theorem 1
b) all $y$-sections $f_y$ continuous, then $C(f)$ is a dense $G_δ$ subset of $(x) \times Y$; thus $C(f)$ is a dense, $G_δ$ subset of $X \times Y$.

In this paper we shall give a shorter proof of the above Theorem 4; rather than using, ad hoc, a Banach-Mazur game, we shall derive this result from some properties of a generalized oscillation function $Ω$.

§4. GENERALIZED OSCILLATION FUNCTION AND QUASI-CONTINUITY.

If $f : X → Y$ is a function and $P$ is an open cover of the space $Y$, then we set:

$Ω(f, P) = \{ x ∈ X : \text{there is an open neighborhood } U \text{ of } x \text{ and a member } V \text{ of } P \text{ such that } f(U) \subset V \}$.

If $Y$ is a metric space and $P_δ$ is the family of all balls of diameter less than $δ$, then $Ω(f, P_δ) = ω(f, P_δ)$ is the set of all points where the oscillation of $f$ does not exceed $δ$.

The sets of the form $Ω(f, P)$ are open.

**Lemma 1.** Let $X$ and $Y$ be spaces and let $\{P_n\}$ be a development for $Z$. If $f : X \times Y → Z$ is quasi-continuous with respect to $x$, then:

$Ω(f, \{P_n\})$ is dense in $(x) \times Y$ for every $x ∈ X$.

**Proof.** Let $x_0 ∈ X$ be arbitrary and let $V$ be a nonempty open subset of $Y$. Now, let $y_0$ be an arbitrary element of $V$. Since $Z$ has a countable development $\{P_n\}$, there is a local countable base at every point of $Z$; in particular, take $\{G_κ\}$ at $f(x_0, y_0)$. Take $G^1$. Now by the quasi-continuity of $f$ with respect to $x$ for every neighborhood $U^* \times V^*$ of $(x_0, y_0)$ (here, we may assume $V^* = V$), there exists a neighborhood $U^1$ of $x$, with $U^1 ⊂ U$ and a nonempty open $V^1 \subset V$ such that $f(U^1 \times V^1) ⊂ G^1$.

Now, let us recall that $Ω(f, \{P_n\}) = \{(x, y) ∈ X \times Y : \text{there is an open neighborhood } U_0 \times V_0 \text{ of } (x, y) \text{ and a member } P \text{ of } P_n \text{ such that } f(U_0 \times V_0) \subset P \}$. Since $\{G_κ\}$ is a local base at $f(x_0, y_0)$ we may assume $P = G^1$. We can also assume $U_0 = U^1$ and $V_0 = V^1$. Clearly, $f(U_0 \times V_0) = f(U^1 \times V^1) ⊂ G^1 = P$.

Further, $Ω(f, \{P_n\}) = U^1 \times V^1$. Observe that $Ω(f, \{P_n\}) \cap (\{x_0\} \times V) = (U^1 \times V^1) \cap (\{x_0\} \times V) = \{x_0\} \times V^1 \neq \emptyset$. Since $V$ is an arbitrary nonempty open subset of $Y$ this shows the density of $Ω(f, \{P_n\})$ in $(x) \times Y$. □

**Lemma 1 [Sz].** Let $\{P_n\}$ be a development for $Y$ and assume $Ω(f, P_n)$ is dense in $X$, for each $n ∈ N$. Then the function $f$ is continuous at each point of a residual subset of $X$.

**Corollary 1.** ([P1], Theorem A) Let $X$ be a space $Y$ be Baire and let $\{P_n\}$ be a development for $Z$. If $f : X \times Y → Z$ is quasi-continuous with respect to $x$, for all $x ∈ X$, then the set

$^4$The original idea is due to A. Szymaniński [Sz], although some similar notions have been used earlier, see [Ew] or [Is].
$C(f)$ of points of continuity of $f$ is residual in the sets of the form $\{x\} \times Y$, i.e. is a dense, $G_δ$ subset in the sets of the form $\{x\} \times Y$.

Now, in view of:

**Lemma 3.** ([LP], Lemma 2) Let $X$ be first countable, $Y$ be Baire and $Z$ regular. If $f : X \times Y \to Z$ has all of its $x$-sections $f_x$ quasi-continuous and has its $y$-sections $f_y$ continuous, with the exception of a first category set, then $f$ is quasi-continuous with respect to $x$.

We have the following statement which slightly generalizes our Theorem 4.

**Statement.** Let $X$ be first countable, $Y$ be Baire and $Z$ regular. If $f : X \times Y \to Z$ has all of its $x$-sections $f_x$ quasi-continuous and has its $y$-sections $f_y$ continuous with the exception of a first category set, then the set $C(f)$ of continuity points of $f$ is a dense $G_δ$ in the sets of the form $\{x\} \times Y$.

§5. **Examples and open questions.**

The following (routine) Example 1 shows that "quasi-continuity with respect to $x$" cannot be replaced by "quasi-continuity" in Corollary 1.

**Example 1.** Let $f : (0,1) \times (0,1) \to \mathbb{R}$ be defined by $f(x,y) = 1$ if $(0 < x < \frac{1}{2} \text{ and } 0 < y < 1)$ or $(x = \frac{1}{2} \text{ and } y \text{ is rational in } (0,1))$, and $f(x,y) = 0$. Clearly, $f$ is quasi-continuous, however there is an $x(= \frac{1}{2})$ such that $C(f) \cap (\{x\} \times Y) = \emptyset$.

**Remark 1.** There is an important link between continuity points in sets of type $\{x\} \times Y$ and some associated mapping.

Namely, any point of the section $\{x\} \times Y$ is a continuity point of $f : X \times Y \to [-1,1]$ if and only if $x$ is a continuity point for the associated mapping:

$F : X \to C(Y)$ defined by

$F(x)(y) = f(x,y)$, with respect to norm-topology on $C(Y)$, see [Ch]; $C(Y)$ stands for the Banach space of continuous functions on $Y$.

The following two Examples show that the assumptions that $X$ is "first countable" and that $Z$ is "Moore" are, in a sense, indispensable.

**Example 2.** ([T1]) Let $Y$ and $Z$ denote the closed unit interval $I = [0,1]$ and let $X$ be the space $C_p(I,I)$ of continuous functions from $I$ into $I$, equipped with he pointwise topology. Then $f : X \times Y \to Z$ given by $f(x,y) = x(y)$ is separately continuous which is discontinuous at every point of $X \times Y$.

It is worth noting that $C_p(I,I)$ is a Tychonoff space having a countable network [E. Michael (1966)] and as such it is hereditarily Lindelöf and hereditarily separable. $C_p(I,I)$

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5See Theorem 4.
is not a Fréchet space and thus not first countable [R. A. McCoy (1980) and J. Gerlits (1983)].

The following Example 3 shows the necessity that the range space \( Z \) is Moore.

**Example 3.** [Ch] Let \( X = Y = [-1, 1] \) and let \( Z \) be the space of mappings from \([-1, 1]^2\) into \([-1, 1]\) equipped with the pointwise topology. Thus \( F(x, y) \) is a function of \((a, b) \in [-1, 1]^2\) given by

\[
F(x, y)(a, b) = \begin{cases} 
\frac{2(x - a)(y - b)}{(x - a)^2 + (y - b)^2}, & \text{if this quotient is defined} \\
0, & \text{otherwise}
\end{cases}
\]

\( F \) is separately continuous, but not (jointly) continuous at any point. \( Z \) is a "large" compact Haussdorff space.

We were unable to answer the following two problems:

**Problem 1.** Let \( X \) be first countable, \( Y \) be Baire and such that \( X \times Y \) is Baire. Given a Moore space (or even metric space) \( Z \). Assume \( f : X \times Y \rightarrow Z \) has:

a) all \( x \)-sections \( f_x \) PWD and;

b) all \( y \)-sections \( f_y \) continuous.

Must \( f \) be PWD, i.e. \( C(f) \) is a dense \( G_\delta \) subset of \( X \times Y \)?

**Problem 2.** Same as Problem 1, except for, assume \( X \) is "compact Haussdorff" instead of "first countable".

**Remark 2.** Answers in positive, to Problem 1 would be strong generalizations of some of the above Theorems.

Symbolically, this would mean: "PWD" \( \times \) "Continuity" = "PWD"

**Remark 3.** Answers in positive, to Problem 2, would solve an outstanding problem of M. Talagrand [T2] whether every real-valued, separately continuous function from a Cartesian product of a Baire space and a compact Haussdorff space has a nonempty set \( C(f) \) of points of continuity.

**REFERENCES**


