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ON SIMULTANEOUS BLUMBERG SETS

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1. Introduction. Throughout the paper a space means a topological space and we do not assume the continuity of functions. For any $A \subset X$, the closure of A and the interior of A are denoted by $\mathrm{Cl}\,A$ and $\mathrm{Int}\,A$, respectively. Given a function $f\colon X \to Y$, denote its set of continuity by $C(f) = \{x \in X \mid f \text{ is continuous at } x\}$.

A function $f\colon X \to Y$ is called *quasi-continuous at a point* $x \in X$ ([7], p. 39) if for any open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$, we have $A \cap \mathrm{Int} f^{-1}(H) \neq \emptyset$. A function $f\colon X \to Y$ is called *quasi-continuous* if it is quasi-continuous at each point x of X.

A function $f\colon X \to Y$ is called *somewhat continuous* if for each open set $V \subset Y$ the condition $f^{-1}(V) \neq \emptyset$ implies $\mathrm{Int} f^{-1}(V) \neq \emptyset$ (see [4], p. 6).

It can be easily verified that any quasi-continuous function is somewhat continuous.

A space X is said to be a *Baire space* ([2], p. 75) if every non-empty open set in X is of second category.

Let f be a function from a space X. We say that X is a *Blumberg space for* f ([11], Definition 3) if there exists a dense subset D of X such that the partial function $f|D$ is continuous. Such a set D is called a *Blumberg set for* f.

A set D is called a *full Blumberg set for* f ([11], Definition 4) if D is a Blumberg set for f and, for every open set $A \subset X$, the set $f(D \cap A)$ is dense in $f(A)$.

Let $f\colon X \to Y$ be a bijection. A set D in X is a *simultaneous Blumberg set for* f ([9], p. 452) if D is a Blumberg set for f and $f(D)$ is a Blumberg set for f^{-1}.

Given a family $F = \{f_i \mid f_i\colon X \to Y \text{ is a bijection}, i \in I\}$, a set D in X is called a *simultaneous Blumberg set for* F if D is a simultaneous Blumberg set for each f_i, $i \in I$.

As the most important results of this paper we consider the Theorem and Corollary 3 in Section 3.

2. Preliminary lemmas and propositions.

LEMMA 1. *Let X be a Baire space, let Y be a second countable space, and let $f\colon X \to Y$ be quasi-continuous. Then $C(f)$ contains a dense G_δ-subset of X.*

The lemma follows easily from Proposition 2 of [3], p. 985.

PROPOSITION 1. *Let X and Y be topological spaces. Let $f\colon X \to Y$ be a somewhat continuous bijection with the somewhat continuous inverse $f^{-1}\colon Y \to X$. If G is a dense subset of X such that $G \subset \mathrm{Cl}\,\mathrm{Int}\,G$, then* $\mathrm{Cl}\,\mathrm{Int}\,f(G) = Y$.

Proof. Let us assume that $\mathrm{Int}\,f(G)$ is not dense in Y. Therefore, there exists a non-empty open set B of Y such that $B \cap \mathrm{Int}\,f(G) = \emptyset$. Then it follows from somewhat continuity of f that $A = \mathrm{Int}\,f^{-1}(B) \neq \emptyset$. Since

$$X = \mathrm{Cl}\,G \subset \mathrm{Cl}\,\mathrm{Cl}\,\mathrm{Int}\,G = \mathrm{Cl}\,\mathrm{Int}\,G,$$

$\mathrm{Int}\,G$ is dense in X. Thus $G' = A \cap \mathrm{Int}\,G \neq \emptyset$. Now, f^{-1} is somewhat continuous, and so $\mathrm{Int}(f^{-1})^{-1}\,G' = \mathrm{Int}\,f(G') \neq \emptyset$. Clearly,

$$\mathrm{Int}\,f(G') = \mathrm{Int}\,f(A \cap \mathrm{Int}\,G) \subset \mathrm{Int}\,f(G).$$

On the other hand,

$$\mathrm{Int}\,f(G') \subset \mathrm{Int}\,f(A) = \mathrm{Int}\,f\big(\mathrm{Int}\,f^{-1}(B)\big) \subset \mathrm{Int}\,f\big(f^{-1}(B)\big) \subset B.$$

Thus we obtain $\emptyset \neq \mathrm{Int}\,f(G') \subset B \cap \mathrm{Int}\,f(G)$, a contradiction.

Note that the set of continuity $C(f)$ of a somewhat continuous function need not be, in general, a dense subset of X (see [11], Remark 1, p. 34). Moreover, a somewhat continuous bijection need not be, in general, quasi-continuous (see [8], Proposition 1, p. 174). However, we have the following

COROLLARY 1. *Let X be a Baire space, let Y be a second countable space, and let $f\colon X \to Y$ be a quasi-continuous bijection with quasi-continuous $f^{-1}\colon Y \to X$. If G is an open subset of X such that G contains a dense subset of $C(f)$, then* $\mathrm{Cl}\,\mathrm{Int}\,f(G) = Y$.

Proof. In fact, by Lemma 1, the set $C(f)$ is dense in X. Thus G is dense in X. Since $G \subset \mathrm{Cl}\,G = \mathrm{Cl}\,\mathrm{Int}\,G$ and since every quasi-continuous function is somewhat continuous, the corollary follows easily from Proposition 1.

LEMMA 2. *If $Q_1, Q_2, \ldots$ are dense G_δ-sets of a Baire space, then so is the set $Q_1 \cap Q_2 \cap \ldots$*

The proof is similar to that of Theorem 1 in [6], § 34, p. 417.

PROPOSITION 2. *Let X and Y be second countable Baire spaces and let F be a countable family of quasi-continuous bijections from X onto Y. If* *for each $f_n \in F$, $n \in N$, the inverse function f_n^{-1} is quasi-continuous, then F admits a simultaneous Blumberg set.*

Proof. By Lemmas 1 and 2, the set $\bigcap_{n=1}^{\infty} C(f_n^{-1})$ contains a dense G_δ-set D of Y. Let $\{G_i\}$ be a sequence of open subsets of Y such that

$$D = \bigcap_{i=1}^{\infty} G_i.$$

Let $\operatorname{Int} f_n^{-1}(G_i) = E_{i,n}$. Then, in virtue of Corollary 1, for all $n \in N$ and for all $i \in N$ the set $E_{i,n}$ is dense in X. Thus

$$E = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} E_{i,n}$$

is a dense G_δ-set of X by Lemma 2.

Arguments similar to those at the beginning of the proof show that $\bigcap_{n=1}^{\infty} C(f_n)$ contains a dense G_δ-set D' of X. Put $H = E \cap D'$. Again, by Lemma 2, H is a dense G_δ-set of X.

To prove that H is a simultaneous Blumberg set for F we assume that f_k is an arbitrary function from F. We have

$$H = E \cap D' \subset D' \subset \bigcap_{n=1}^{\infty} C(f_n) \subset C(f_k),$$

which shows that H is a Blumberg set for f_k. Further, we obtain

$$\begin{aligned} f_k(H) &= f_k\Big(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \operatorname{Int} f_n^{-1}(G_i) \cap D'\Big) \subset f_k\Big(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \operatorname{Int} f_n^{-1}(G_i)\Big) \\ &\subset f_k\Big(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} f_n^{-1}(G_i)\Big) \subset f_k\Big(\bigcap_{i=1}^{\infty} f_k^{-1}(G_i)\Big) = f_k\Big(f_k^{-1}\Big(\bigcap_{i=1}^{\infty} G_i\Big)\Big) \\ &= \bigcap_{i=1}^{\infty} G_i = D \subset \bigcap_{n=1}^{\infty} C(f_n^{-1}) \subset C(f_k^{-1}). \end{aligned}$$

This shows that $f_k(H)$ is a Blumberg set for f_k^{-1}. Thus the proof is completed.

By our method we see that a simultaneous Blumberg set for F is a G_δ-subset of X. This generalizes some results of Neugebauer (see [9], p. 452).

The countability of F is essential.

Example 1. Consider an uncountable family F^* of quasi-continuous bijections f_a of $[0, 1]$. Given $a \in (0, 1/2]$, define f_a as follows:

for $a \in (0, 1/2)$,

$$f_a(x) = \begin{cases} x & \text{for } x \in [0, a] \cup [1-a, 1], \\ -x+1 & \text{for } x \in (a, 1-a); \end{cases}$$

for $a = 1/2$,

$$f_{1/2}(x) = \begin{cases} x & \text{for } x \in [0, 1/2), \\ -x+3/2 & \text{for } x \in [1/2, 1]. \end{cases}$$

There exists no simultaneous Blumberg set for F^*, since every point $x_0 \in (0, 1)$ is a point of discontinuity of a function of the family F^*, namely f_{x_0} if $x_0 \leqslant 1/2$, or f_{1-x_0} if $x_0 > 1/2$.

PROPOSITION 3. *Let $f\colon X \to Y$ be a quasi-continuous bijection. If D is a simultaneous Blumberg set for f, then $f(D)$ is a full Blumberg set for f^{-1}.*

Proof. Put $D' = f(D)$. We will show that for each open subset J of Y the set $f^{-1}(D' \cap J)$ is dense in $f^{-1}(J)$. Take an open subset K of X such that $K \cap f^{-1}(J) \neq \emptyset$. If $x_0 \in K \cap f^{-1}(J)$, then $f(x_0) \in J$. Since f is quasi-continuous at x_0, there exists a non-empty open set $U \subset K$ such that $f(U) \subset J$. The density of D in X implies $U \cap D \neq \emptyset$. But $U \subset K$ and $U \subset f^{-1}(J)$. Therefore

$$\emptyset \neq U \cap D = U \cap f^{-1}\big(f(D)\big) = U \cap f^{-1}\big(f(D) \cap J\big) \subset K \cap f^{-1}(D' \cap J).$$

Thus $K \cap f^{-1}(D' \cap J) \neq \emptyset$.

From Proposition 3 and Theorem 2 of [11] we obtain

COROLLARY 2. *Let $f\colon X \to Y$ be a quasi-continuous bijection, where X is a regular space, and Y is a Blumberg space for f^{-1}. If D is a simultaneous Blumberg set for f, then f^{-1} is quasi-continuous.*

Proof. In fact, $f\colon X \to Y$ is a quasi-continuous bijection and D is a simultaneous Blumberg set for f. Thus, by Proposition 3, there exists a full Blumberg set for f^{-1}. Now, f^{-1} is a function from a space Y, which is a Blumberg space for f^{-1}, into a regular space X. Hence Theorem 2 of [11], p. 34, can be applied, and thereby f^{-1} is quasi-continuous.

The author is indebted very much to the reviewer for the following example showing that the regularity of X is essential in Corollary 2.

Example 2. Take the reals with the natural topology as Y. As X take the reals with the topology which is finer than the natural topology by assuming the set of irrationals to be open. The identity function from X onto Y admits a simultaneous Blumberg set (namely, the set of irrationals), but its inverse function is not quasi-continuous. The fact that Y is a Blumberg space for f^{-1} follows easily from Alas' statement quoted in Section 3.

3. Main theorem.

THEOREM. *Let X and Y be second countable Baire spaces, let X be regular, let F be a countable family of quasi-continuous bijections from X onto Y, and let Y be a Blumberg space for f_n^{-1}, for every $f_n \in F$. Then F admits a simultaneous Blumberg set if and only if for every $f_n \in F$ the inverse function f_n^{-1} is quasi-continuous.*

The Theorem follows easily from Proposition 2 and Corollary 2.

There exists an example ([9], Theorem 3, p. 454) of a function from [0, 1] onto itself which is a quasi-continuous bijection and whose inverse is not quasi-continuous. Another one-to-one function which does not admit a simultaneous Blumberg set was given by Goffman [5].

Now we recall two definitions and a result due to Alas [1].

A *pseudobase* ([10], p. 157) for a space X with the topology T is a subset P of T such that every non-empty element of T contains a non-empty element of P.

A subfamily P of T is called *σ-disjoint* ([12], p. 456) if

$$P = \bigcup \{P_n\colon n = 1, 2, \dots\},$$

where each P_n is a disjoint family.

STATEMENT (Alas). *Let X be a Hausdorff, Baire space with a σ-disjoint pseudobase, let Y be a Hausdorff second countable space, and let $f\colon X \to Y$ be a function. There exists a dense subset D of X such that the restriction of f to D is continuous.*

Every second countable space has a σ-disjoint pseudobase. Therefore, if X is a Hausdorff, Baire, second countable space, Y is a Hausdorff second countable space, and $f\colon X \to Y$ is a function, then X is a Blumberg space for f. Thus we have a result which follows from the Theorem and Alas' statement:

COROLLARY 3. *Let X and Y be second countable, Hausdorff, Baire spaces, let X be regular, and let F be a countable family of quasi-continuous bijections from X onto Y. Then F admits a simultaneous Blumberg set if and only if for each $f_n \in F$, $n \in N$, the inverse function f_n^{-1} is quasi-continuous.*

COROLLARY 4 ([9], Theorem 2, p. 452). *Let f be a quasi-continuous bijection from the unit interval onto itself. Then f admits a simultaneous Blumberg set if and only if f^{-1} is quasi-continuous.*

PROBLEM (P 1234). Does the Theorem remain true if the requirements on X or Y to be Baire spaces are omitted? I conjecture that the answer is negative.

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