Real Analysis Exchange
Vol. 22(1), 1996/7, pp. 87–88

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ON SOME PROBLEMS ON SEPARATE VERSUS JOINT CONTINUITY

Let X, Y and M be "nice" spaces and let $f : X \times Y \to M$ be a function.

Firstly, we shall deal with the question pertaining to the existence of the continuity points $C(f)$ under various assumptions pertaining to the x-sections f_x and y-sections f_y.

Notice that although Baire-Lebesgue-Kuratowski-Montgomery theorems "handles well" the case when f is separately continuous – f is of 1st class then (see W. Rudin (1981), Moran (1969) and M. Henriksen, G. Woods (preprint)), Baire classification of functions is "too rough" already in the case when all x-sections are continuous and all y-sections are of 1st class – f is of 2nd class then.

Consider the following statement:

> (*) Given a metric space M. Let $X \times Y$ be a Baire space and let $f : X \times Y \to M$ be a function having all y-sections continuous. Then $C(f)$ is a dense G_δ subset of $X \times Y$.

Y. Mibu (1958) showed that (*) holds, if X is 1st countable and f is separately continuous. He proved also that (*) is true when X is 2nd countable and f has all x-sections pointwise discontinuous.

G. Debs (1987) showed that (*) holds if X is 1st countable, Y is a "special" α-favorable (hence, Baire) and f has all of its x-sections of the 1st class (in his sense). The author [(1993) and (1996) – for an alternative proof)] showed that (*) is valid if X is 1st countable Y-Baire and M-Moore and $f : X \times Y \to M$ has all x-sections quasi-continuous.

Problem 1 *Let X be 1st countable and let $f : X \times Y \to M$ has all x-sections pointwise discontinuous. Does (*) hold?*

What if Y is Čech-complete?

* * *

Assume that X is Čech-complete, Y is locally compact and σ-compact and Z is metric. Assume $f : X \times Y \to Z$ is separately continuous I. Namioka (1974) showed that then there is a dense G_δ set $A \subset X$ such that $A \times Y \subset C(f)$.

M. Talagrand (1985) asked the following problem: Let X be Baire, Y be compact, Hausdorff and let $f : X \times Y \to I\!R$ be separately continuous. Is $C(f)$ nonempty?

Recall that a function $f : X \to Y$ is termed *feebly continuous* if $\forall V \subset Y$: $f^{-1}(V) \neq \phi \Rightarrow Int f^{-1}(V) \neq \phi$.

Theorem 1 (E. J. Wingler and the author – "Q & A in General Topology," (accepted)) *Assume that every separately continuous function f from the product $f : X \times Y$ into a completely regular space is feebly continuous. Then any separately continuous function from $f : X \times Y$ into Z is determined by its values on any dense subset of the domain.*

Problem 2 *Let X be a Baire space and let Y be compact T_2. Is every separately continuous function $f : X \times Y \to \mathbb{R}$ feebly continuous?*

Remark 1 *A positive answer to Problem 2 would solve Talagrand's problem, since such a feebly continuous function defined on a Baire space $(X \times Y)$ has $C(f)$ nonempty.*

* * *

R. Kershner (1944) characterized the set $C(f)$ of a separately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$. Namely, if $X = Y = \mathbb{R}$:

> ($^*_*$) Let $S \subset X \times Y$. Then $X = Y \times Y \backslash C(f)$ of a separately continuous function $f : X \times Y \to I\!R$ if and only if S is an F_σ contained in the product of two sets of 1st category.

J. C. Breckenridge and T. Nishiura have generalized this result to compact metric spaces X, Y (1976).

Answering the author's question (1989), V. K. Maslyuchenko, V. V. Mykhaylyuk and O. V. Sobchuk (1992) showed that Kershner-Breckenridge-Nishiura's characterization is no longer true, if X and Y are arbitrary compact, Hausdorff spaces.

Problem 3 *Find the largest class $\mathcal{P}$ of metric spaces such as $X, Y \in \mathcal{P}$ if and only if ($^*_*$) holds.*

Are UC (known also to Atsuji, or Lebesgue) spaces the spaces for which ($^*_*$) holds?

Real Analysis Exchange
Vol. 22(1), 1996/7, pp. 89–90

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WHY IS SYMMETRIC POROSITY SO DIFFERENT?

This talk was based on joint work [2] with Paul Humke.

Porous sets and symmetrically porous sets have previously been contrasted in [6], [3], [4], [5] and [8]. Both of [6] and [3] pointed out that the following two fundamental properties of porosity fail for symmetric porosity: 1) [1] *Every nowhere dense set A contains a residual subset of points x at which $p(A, x) = 1$.* 2) [7] *If A is a porous set and $0 < p < 1$, then A can be written as a countable union of p-porous sets.* For example, in [3] a closed 1/2-symmetrically porous set A with the property that $\mathrm{sp}(A, x) \leq 4/5$ for every $x \in A$ was exhibited, and it was observed that such a set cannot be written as a countable union of sets having symmetric porosity more than 4/5 at each of their points. We take such results as the starting point for the present investigation [2] to explore such questions as

i. If E is a p-symmetrically porous set, must there be any points in E having symmetric porosity greater than p? (If so, is the collection of such points residual in E and how large can the symmetric porosity at such points be?)

ii. If E is a p-symmetrically porous set, can E be written as a countable union of sets, each of which has symmetric porosity greater than p at each of its points? (If so, can we find a $q > p$ such that each of the constituent sets is q-symmetrically porous?)

Our results include the following:

Theorem 1 *If $0 < p < 1$ and E is a closed set which has symmetric porosity at least p at each of its points, then there exists a number q, $p < q < 1$, such that the set*

$$\{x \in E : \textit{the symmetric porosity of } E \textit{ at } x \textit{ is at least } q\}$$

is residual in E.

Example 1 *Given $0 < p < 1$, there exists a G_δ set $E \subseteq [0, 1]$ such that E has symmetric porosity exactly p at each of its points.*

Example 2 *Given $0 < p < 1$, there exists a closed set E, such that E which has symmetric porosity at least p at each of its points, but cannot be written as the countable union of sets each of which has symmetric porosity greater than p at each of its points.*

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