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ON SOME PROBLEMS ON SEPARATE 
VERSUS JOINT CONTINUITY

Let $X, Y$ and $M$ be “nice” spaces and let $f : X \times Y \to M$ be a function. 
Firstly, we shall deal with the question pertaining to the existence of the 
continuity points $C(f)$ under various assumptions pertaining to the $x$-sections 
$f_x$ and $y$-sections $f_y$. 

Notice that although Baire-Lebesgue-Kuratowski-Montgomery theorems 
“handles well” the case when $f$ is separately continuous – $f$ is of 1st class then 
(see W. Rudin (1981), Moran (1969) and M. Henriksen, G. Woods (preprint)), 
Baire classification of functions is “too rough” already in the case when all $x$-
sections are continuous and all $y$-sections are of 1st class – $f$ is of 2nd class 
then.

Consider the following statement:

(*) Given a metric space $M$. Let $X \times Y$ be a Baire space and let 
$f : X \times Y \to M$ be a function having all $y$-sections continuous. 
Then $C(f)$ is a dense $G_\delta$ subset of $X \times Y$.

Y. Mibu (1958) showed that (*) holds, if $X$ is 1st countable and $f$ is 
separately continuous. He proved also that (*) is true when $X$ is 2nd countable 
and $f$ has all $x$-sections pointwise discontinuous.

G. Debs (1987) showed that (*) holds if $X$ is 1st countable, $Y$ is a “special” 
$\alpha$-favorable (hence, Baire) and $f$ has all of its $x$-sections of the 1st class (in his 
sense). The author [(1993) and (1996) – for an alternative proof)] showed that 
(*) is valid if $X$ is 1st countable $Y$-Baire and $M$-Moore and $f : X \times Y \to M$ 
has all $x$-sections quasi-continuous.

Problem 1 Let $X$ be 1st countable and let $f : X \times Y \to M$ has all $x$-sections 
pointwise discontinuous. Does (*) hold?

What if $Y$ is Čech-complete?

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Assume that $X$ is Čech-complete, $Y$ is locally compact and $\sigma$-compact and $Z$ is metric. Assume $f : X \times Y \to Z$ is separately continuous. I. Namioka (1974) showed that then there is a dense $G_\delta$ set $A \subset X$ such that $A \times Y \subset C(f)$.

M. Talagrand (1985) asked the following problem: Let $X$ be Baire, $Y$ be compact, Hausdorff and let $f : X \times Y \to \mathbb{R}$ be separately continuous. Is $C(f)$ nonempty?

Recall that a function $f : X \to Y$ is termed **feebly continuous** if $\forall V \subset Y : f^{-1}(V) \neq \emptyset \Rightarrow Int f^{-1}(V) \neq \emptyset$.

**Theorem 1** (E. J. Wingler and the author – “Q & A in General Topology,” (accepted)) Assume that every separately continuous function $f$ from the product $f : X \times Y$ into a completely regular space is feebly continuous. Then any separately continuous function from $f : X \times Y$ into $Z$ is determined by its values on any dense subset of the domain.

**Problem 2** Let $X$ be a Baire space and let $Y$ be compact $T_2$. Is every separately continuous function $f : X \times Y \to \mathbb{R}$ feebly continuous?

**Remark 1** A positive answer to Problem 2 would solve Talagrand’s problem, since such a feebly continuous function defined on a Baire space $(X \times Y)$ has $C(f)$ nonempty.

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R. Kershner (1944) characterized the set $C(f)$ of a separately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$. Namely, if $X = Y = \mathbb{P}$:

\[ (*) \text{ Let } S \subset X \times Y. \text{ Then } X = Y \times Y \setminus C(f) \text{ of a separately continuous function } f : X \times Y \to \mathbb{R} \text{ if and only if } S \text{ is an } F_\sigma \text{ contained in the product of two sets of 1st category.} \]

J. C. Breckenridge and T. Nishiura have generalized this result to compact metric spaces $X, Y$ (1976).

Answering the author’s question (1989), V. K. Maslyuchenko, V. V. Mykhajlyuk and O. V. Sobchuk (1992) showed that Kershner-Breckenridge-Nishiura’s characterization is no longer true, if $X$ and $Y$ are arbitrary compact, Hausdorff spaces.

**Problem 3** Find the largest class $\mathcal{P}$ of metric spaces such as $X, Y \in \mathcal{P}$ if and only if $(\star)$ holds.

Are $\mathcal{U}C$ (known also to Atsuji, or Lebesgue) spaces the spaces for which $(\star)$ holds?
WHY IS SYMMETRIC POROSITY SO DIFFERENT?

This talk was based on joint work [2] with Paul Humke.

Porous sets and symmetrically porous sets have previously been contrasted in [6], [3], [4], [5] and [8]. Both of [6] and [3] pointed out that the following two fundamental properties of porosity fail for symmetric porosity: 1) [1] Every nowhere dense set $A$ contains a residual subset of points $x$ at which $p(A, x) = 1$. 2) [7] If $A$ is a porous set and $0 < p < 1$, then $A$ can be written as a countable union of $p$-porous sets. For example, in [3] a closed 1/2-symmetrically porous set $A$ with the property that $sp(A, x) \leq 4/5$ for every $x \in A$ was exhibited, and it was observed that such a set cannot be written as a countable union of sets having symmetric porosity more than 4/5 at each of their points. We take such results as the starting point for the present investigation [2] to explore such questions as

i. If $E$ is a $p$-symmetrically porous set, must there be any points in $E$ having symmetric porosity greater than $p$? (If so, is the collection of such points residual in $E$ and how large can the symmetric porosity at such points be?)

ii. If $E$ is a $p$-symmetrically porous set, can $E$ be written as a countable union of sets, each of which has symmetric porosity greater than $p$ at each of its points? (If so, can we find a $q > p$ such that each of the constituent sets is $q$-symmetrically porous?)

Our results include the following:

**Theorem 1** If $0 < p < 1$ and $E$ is a closed set which has symmetric porosity at least $p$ at each of its points, then there exists a number $q$, $p < q < 1$, such that the set

$$ \{ x \in E : \text{the symmetric porosity of } E \text{ at } x \text{ is at least } q \} $$

is residual in $E$. 89
Example 1 Given $0 < p < 1$, there exists a $G_{δ}$ set $E \subseteq [0, 1]$ such that $E$ has symmetric porosity exactly $p$ at each of its points.

Example 2 Given $0 < p < 1$, there exists a closed set $E$, such that $E$ which has symmetric porosity at least $p$ at each of its points, but cannot be written as the countable union of sets each of which has symmetric porosity greater than $p$ at each of its points.

References


