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J. Math. Anal. Appl. 313 (2006) 38–48
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Plasticity in metric spaces

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Received 21 March 2003

Available online 25 May 2005

Submitted by B.S. Thomson

Abstract

In this paper we examine the properties of EC-plastic metric spaces, spaces which have the property that any noncontractive bijection from the space onto itself must be an isometry.

Keywords: Compact metric space; Plastic metric space; Hippopotamus

1. Background

To motivate the definition of a plastic space, we begin by stating a theorem and its corollary. These are similar to some of the theorems found in [2].

Theorem 1.1. *Let (X, d) be a totally bounded metric space, and let $f : X \to X$ be a function. If there exist points p and q such that $d(f(p), f(q)) > d(p, q)$, then there exist points r and s such that $d(f(r), f(s)) < d(r, s)$.*

Proof. Suppose that such points r and s do not exist. Then $d(f(x), f(y)) \geqslant d(x, y)$ for all $x, y \in X$. In particular, $d(f^n(p), f^n(q)) \geqslant d(p, q)$ for all positive integers n. Since

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.04.070

X is totally bounded, the sequences $(f^n(p))$ and $(f^n(q))$ contain Cauchy subsequences $(f^{n_k}(p))$ and $(f^{n_k}(q))$.

Let $\varepsilon > 0$. Then there exists a number k such that for all $j \geqslant 1$,

$$d\big(f^{n_{k+j}}(p), f^{n_k}(p)\big) < \frac{\varepsilon}{2} \quad \text{and} \quad d\big(f^{n_{k+j}}(q), f^{n_k}(q)\big) < \frac{\varepsilon}{2}.$$

Hence

$$d\big(f^{n_{k+j}-n_k}(p), p\big) \leqslant d\big(f^{n_{k+j}}(p), f^{n_k}(p)\big) < \frac{\varepsilon}{2}$$

and

$$d\big(f^{n_{k+j}-n_k}(q), q\big) \leqslant d\big(f^{n_{k+j}}(q), f^{n_k}(q)\big) < \frac{\varepsilon}{2}$$

for all $j \geqslant 1$. So

$$d\big(f(p), f(q)\big) \leqslant d\big(f^{n_{k+j}-n_k}(p), f^{n_{k+j}-n_k}(q)\big) < d(p,q) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $d(f(p), f(q)) \leqslant d(p,q)$, which contradicts the hypothesis. $\square$

Notice that in the preceding theorem there are no restrictions on the function f. It is not required to be continuous, injective, or surjective. Satz IV of [2] states that a mapping of a totally bounded space onto itself is either an isometry or there will be a pair of points whose distance increases under this mapping and another pair of points whose distance decreases. This theorem requires the surjectivity of the mapping, but on the other hand, it goes beyond Theorem 1.1 in that the decreasing of the distance between a pair of points implies the increasing of the distance between another pair of points.

Corollary 1.2. *Let X be a compact metric space and let $f : X \to X$ be noncontractive, that is, $d(f(p), f(q)) \geqslant d(p,q)$ for all $p, q \in X$. Then f is an isometry onto X.*

Proof. Since there do not exist points r and s such that $d(f(r), f(s)) < d(r,s)$, it follows from Theorem 1.1 that there are no points p and q such that $d(f(p), f(q)) > d(p,q)$. Hence f is an isometry. Since a compact space cannot be isometric to a proper subspace of itself (see [3, p. 194]), f must be surjective. $\square$

In Corollary 1.2 if "compact" is replaced with "totally bounded," f will still be an isometry, but it may not be surjective as the following example demonstrates.

Example 1.3. Let $X = \{e^{in}\colon n \in \mathbf{N}\}$, where $\mathbf{N}$ denotes the set of positive integers, and let $f(z) = e^i z$. Since f is a rotation of the complex plane, it is an isometry onto $f(X)$. However, f is not surjective because e^i is not in its range.

In the example above, f is not surjective, but $f(X)$ is dense in X. This is not a coincidence; it is a theorem of A. Lindenbaum (see [1, 4.3.D] or [5]) that if X is totally bounded and $f : X \to X$ is an isometry, then $f(X)$ is dense in X.

Recently, Nitka in [7] has quantified the compensation made for the contraction of the distance between two points. Specifically, let M be any totally bounded metric space and let z be any positive real number. Then there exists $r > 0$ such that for every surjection $f: M \to M$ if there exist $a, b \in M$ such that $d(f(a), f(b)) < d(a,b) - z$, then there exist $p, q \in M$ such that $d(p,q) + r < d(f(p), f(q))$. In particular, r can be chosen to be equal to $2\varepsilon/(n_\varepsilon(n_\varepsilon - 1) + 2)$, where $\varepsilon = z/11$ and n_ε is the number of elements in an ε-net in M.

2. Definition of a plastic space

Consider the following three properties that a metric space X might have, where $f(x)$ is denoted by x' for any $x \in X$:

(A) For each surjection $f: X \to X$ if there exist $a, b \in X$ such that $d(a', b') > d(a, b)$, then there exist $p, q \in X$ such that $d(p', q') < d(p, q)$.
(B) For each surjection $f: X \to X$ if there exist $a, b \in X$ such that $d(a', b') < d(a, b)$, then there exist $p, q \in X$ such that $d(p', q') > d(p, q)$.
(C) For each surjection $f: X \to X$ if f is noncontractive, then f is an isometry.

It is an easy exercise to show that (A) and (C) are equivalent, and either of these properties is implied by (B). Property (B) is not equivalent to the other two however. As will be shown later, the set **Z** of integers with the absolute value metric satisfies property (A). However it does not satisfy property (B). For example, consider the function $f: \mathbf{Z} \to \mathbf{Z}$ defined by

$$f(n) = \begin{cases} n+1, & \text{if } n < 0, \\ n, & \text{if } n \geqslant 0. \end{cases}$$

Notice that $|f(-1) - f(0)| = 0 < |-1 - 0|$, but in no case do we have $|f(p) - f(q)| > |p - q|$.

Since a noncontractive mapping is automatically injective, the word "surjection" in property (C) can replaced by "bijection" without changing the meaning of (C). It is this modification of property (C) that we will use in our definition for an Expand-Contract plastic space.

Definition 2.1. A metric space X is called an *Expand–Contract plastic space* (or simply, an *EC-space*) if every noncontractive bijection from X onto itself is an isometry. Metric spaces that are not EC-spaces will be called *NEC-spaces.* A metric space X is called a *Contract–Expand plastic space* (or *CE-space*) if every nonexpansive surjection from X onto itself is an isometry.[1]

Note. Since the inverse of a noncontractive bijection is a nonexpansive bijection, it follows that if every nonexpansive bijection from X onto itself is an isometry, then X is an EC-space.

[1] Because of a remark by B. Knaster, these spaces came to be called hippopotamus spaces. Apparently they were given this name because some employees at the local zoo had remarked that the skin of a hippopotamus is so tight that when it contracts in one area, it expands in another (see [7]).

By Theorem 1.1, every totally bounded space is an EC-space. Also, every metric space with the 0–1 metric is EC-plastic simply because every bijection is an isometry. An example similar to this last one is the following: let X be an infinite set containing the distinct elements a and b and let d be the metric defined by

$$d(x,y)=\begin{cases}2, & \text{if } \{x,y\}=\{a,b\};\\ 0, & \text{if } x=y;\\ 1, & \text{if } \{x,y\}\neq\{a,b\} \text{ and } x\neq y.\end{cases}$$

Then X is an EC-space.

EC-spaces need not be locally compact as the example of the rationals in $[0,1]$ shows.

3. Set theoretic properties

In this section we will investigate various properties of both EC- and NEC-spaces. So far every example of an EC-space we have given, with one exception, has been bounded. The exception is $(\mathbf{Z}, |\cdot|)$.

Theorem 3.1. *The set* $\mathbf{Z}$ *of integers with the usual metric is an EC-space.*

Proof. Let $f:\mathbf{Z}\to\mathbf{Z}$ be a noncontractive bijection. Suppose there are integers p and q such that $|f(p)-f(q)|>|p-q|$. Let $r=|p-q|+1$. The open ball $B(f(p),r)$ contains $2r-1$ elements. Since f is noncontractive, $f^{-1}(B(f(p),r))\subseteq B(p,r)$. But q is in $B(p,r)$, while $f(q)$ is not in $B(f(p),r)$. Hence $f^{-1}(B(f(p),r))$ contains fewer than $2r-1$ elements, which is impossible since f is a bijection. Therefore f is an isometry. □

This theorem also answers the question of whether the EC-plastic property is hereditary. Consider the subset S of $\mathbf{Z}$ defined by $S=\{n\colon n<0\}\cup\{2n\colon n\geqslant 0\}$. The function $f:S\to S$, defined by

$$f(n)=\begin{cases}n+1, & \text{if } n<0,\\ n+2, & \text{if } n=2k\geqslant 0 \text{ for some } k\in\mathbf{Z},\end{cases}$$

is a noncontractive bijection, but it is not an isometry. Hence, S is not an EC-space; consequently, the EC-plastic property is not hereditary. The NEC-plastic property is not hereditary either. For example, $(\mathbf{R}, |\cdot|)$ is an NEC-space, whereas $[0,1]$ is not.

Neither the EC-plastic property nor the NEC-plastic property are necessarily preserved by unions. Let $S=\{-n\colon n\in\mathbf{N}\}$ and $T=\{2n-1\colon n\in\mathbf{N}\}$. Both of these subspaces of $\mathbf{Z}$ can be shown to be EC-spaces using an argument similar to that used in the proof of Theorem 3.1. However, $S\cup T$ is an NEC-space. Let $P=\{2-2n\colon n\in\mathbf{N}\}\cup\{3n\colon n\in\mathbf{N}\}$. Then both P and $\mathbf{Z}-P$ are NEC-spaces, but of course, their union is an EC-space. Is the intersection of two NEC-spaces an NEC-space? Not necessarily. Both $(-\infty,1)$ and $(0,\infty)$ are NEC-spaces, but their intersection, $(0,1)$, is an EC-space. At the end of this section, we will show that the intersection of two EC-spaces need not be an EC-plastic space.

The next set of theorems will be used to answer questions about products of EC-spaces and the completion of an NEC-space. We will use the following theorem of Sierpiński (see [1, p. 440]) in the proof of our next theorem.

Theorem 3.2. *If a compact connected space X has a countable cover $\{X_i\}_{i=1}^{\infty}$ by pairwise disjoint closed subsets, then at most one of the sets X_i is nonempty.*

Theorem 3.3. *Let (K,d) be a connected, compact, metric space and let $\mathbf{Z}$ be the set of integers with the absolute value metric. Then $K \times \mathbf{Z}$ (endowed with the usual product metric ρ) is an EC-space.*

Proof. Let $f: K \times \mathbf{Z} \to K \times \mathbf{Z}$ be a noncontractive bijection. Then f^{-1} is nonexpansive, so it is continuous. If n is any integer, then $K \times \{n\}$ is connected. Hence $f^{-1}(K \times \{n\}) \subseteq K \times \{m_n\}$ for some integer m_n. In fact, we will show that for each integer n there is a unique integer n^* such that $f(K \times \{n\}) = K \times \{n^*\}$, and the mapping $n \mapsto n^*$ is a bijection.

Let n be an integer, and let

$$S = \{p \in \mathbf{Z}\colon\ f^{-1}(K \times \{p\}) \subseteq K \times \{m_n\}\}.$$

Then $f^{-1}(K \times S) \subseteq K \times \{m_n\}$. Let $x \in K$ and let $f(x, m_n) = (y, p)$. Then

$$(x, m_n) \in f^{-1}(K \times \{p\}) \subseteq K \times \{m_p\},$$

which implies that $m_n = m_p$. Hence $p \in S$; so $f(K \times \{m_n\}) = K \times S$. Now, this implies that $K \times \{m_n\} = f^{-1}(K \times S)$ is the union of a countable collection of disjoint closed sets. Since, of course, we may suppose that $K \neq \emptyset$, it follows from Sierpiński's theorem that $\operatorname{card}(S) = 1$. Hence $S = \{n\}$. This shows that $f(K \times \{m_n\}) = K \times \{n\}$.

We have now established the following: for each integer n there is an integer n^* and a noncontractive bijection $g_n: K \to K$ such that $f(x, n) = (g_n(x), n^*)$ and the mapping $n \mapsto n^*$ is a bijection on $\mathbf{Z}$. Since K is an EC-space, the mapping g_n must be an isometry for each n. We will now show that the mapping $n \mapsto n^*$ is an isometry as well.

Suppose this is not the case. Then since $\mathbf{Z}$ is an EC-space, there must exist $m, n \in \mathbf{Z}$ with $|m^* - n^*| < |m - n|$. Fix an element $x \in K$ and let $y, z \in K$ be such that $g_m(y) = g_n(z) = x$. Then we have that

$$\begin{aligned}
\rho\big(f(y,m), f(z,n)\big) &= \rho\big(\big(g_m(y), m^*\big), \big(g_n(z), n^*\big)\big) = \rho\big((x, m^*), (x, n^*)\big) \\
&= \sqrt{\big(d(x,x)\big)^2 + |m^* - n^*|^2} \\
&= |m^* - n^*| < |m - n| = \sqrt{|m-n|^2} \\
&\leqslant \sqrt{\big(d(y,z)\big)^2 + |m-n|^2} = \rho\big((y,m), (z,n)\big).
\end{aligned}$$

In the final step of the proof, we will show that the isometries g_n are identical. Let m and n be distinct integers and suppose that $g_n \neq g_m$. Then there is an element $x \in K$ such that $g_n(x) \neq g_m(x)$. Let $y = g_n^{-1}(g_m(x))$. Then

$$\begin{aligned}
\rho\big(f(y,n), f(x,m)\big) &= \big[d\big(g_n(y), g_m(x)\big)^2 + |n^* - m^*|^2\big]^{1/2} \\
&= |n^* - m^*| = |n - m| < \big[d(y,x)^2 + |n-m|^2\big]^{1/2} \\
&= \rho\big((y,n), (x,m)\big),
\end{aligned}$$

which is impossible since f is noncontractive. Hence $g_n = g_m$, and letting g denote g_n, we have $f(x,n) = (g(x), n^*)$ for all $x \in K$ and $n \in \mathbf{Z}$. Clearly f is an isometry, so $K \times \mathbf{Z}$ is an EC-space. □

Note. The preceding theorem holds if the metric ρ is replaced with the supremum metric ρ' defined by

$$\rho'\big((x,m),(y,n)\big) = \max\big\{d(x,y), |m-n|\big\}.$$

Corollary 3.4. $[0,1] \times \mathbf{Z}$ *is an EC-space.*

Note. In a manner similar to that used in the above theorem, it can be shown that if D is an infinite discrete space with the 0–1 metric, then $[0,1] \times D$ is an EC-space. This is another example of a noncompact, dense-in-itself, complete EC-space.

A question that arises naturally is whether the hypotheses can be weakened to requiring that K be totally bounded rather than compact. The answer is no.

Theorem 3.5. $[0,1) \times \mathbf{Z}$ *is an NEC-space.*

Proof. Let

$$f(x,n) = \begin{cases} (2x, 2n), & \text{if } 0 \leqslant x < \frac{1}{2} \text{ and } n \text{ is even;} \\ (2x-1, 2n+1), & \text{if } \frac{1}{2} \leqslant x < 1 \text{ and } n \text{ is even;} \\ (2x, 2n+1), & \text{if } 0 \leqslant x < \frac{1}{2} \text{ and } n \text{ is odd;} \\ (2x-1, 2n), & \text{if } \frac{1}{2} \leqslant x < 1 \text{ and } n \text{ is odd.} \end{cases}$$

It can be verified easily that f is a noncontractive bijection that is not an isometry. The mapping f doubles the lengths of parallel line segments and maps them onto parallel line segments in such a way that the distance between segment images is never less than the distance between the original segments. Moreover, whenever (x,n) and (y,m) are such that, letting $f(x,n) = (z,h)$ and $f(y,m) = (w,k)$, we have $|z-w| < |x-y|$, then we also have that $|h-k| \geqslant |n-m| + 1$ (see Fig. 1). □

At this point we see two things: the product of two EC-spaces need not be an EC-space, and the completion of an NEC-space need not be an NEC-space. In the theorem above can the hypothesis that K is connected be removed? No.

Theorem 3.6. *If K is the Cantor set, then $K \times \mathbf{Z}$ is an NEC-space.*

Proof. Let

$$f(x,n) = \begin{cases} (3x, 2n), & \text{if } 0 \leqslant x \leqslant \frac{1}{3} \text{ and } n \text{ is even;} \\ (3x-2, 2n+1), & \text{if } \frac{2}{3} \leqslant x \leqslant 1 \text{ and } n \text{ is even;} \\ (3x, 2n+1), & \text{if } 0 \leqslant x \leqslant \frac{1}{3} \text{ and } n \text{ is odd;} \\ (3x-2, 2n), & \text{if } \frac{2}{3} \leqslant x \leqslant 1 \text{ and } n \text{ is odd.} \end{cases}$$

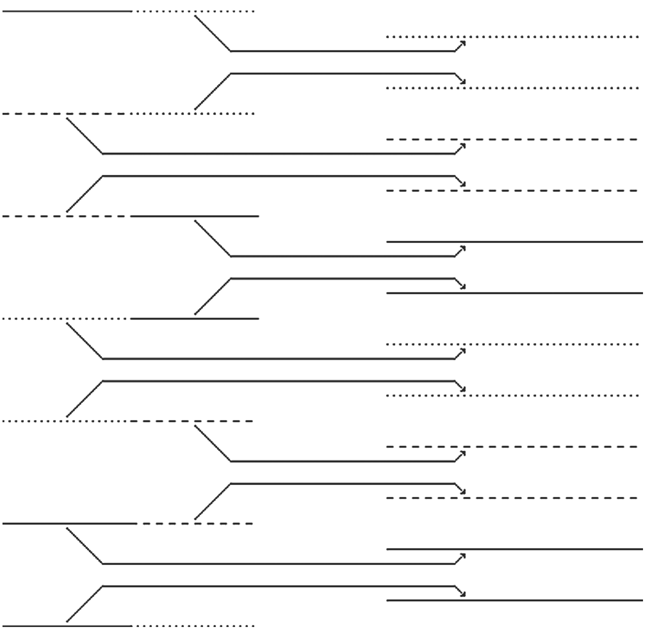

Fig. 1. The mapping $f : [0, 1) \times \mathbf{Z} \to [0, 1) \times \mathbf{Z}$.

It can be verified easily that f is a noncontractive bijection that is not an isometry. Here we have made use of the fact that the sets $K \cap [0, 1/3]$ and $K \cap [2/3, 1]$ are similar to K. □

This theorem demonstrates that the product of two EC-spaces need not be an EC-space, even if one of the factors is compact.

Theorem 3.7. *If X is an NEC-space, then $X \times Y$ is an NEC-space for any metric space Y.*

Proof. Since X is an NEC-space, there exists a noncontractive bijection $f : X \to X$ that is not an isometry. Let g be the identity map on Y. Then $f \times g : X \times Y \to X \times Y$ is again a noncontractive bijection that is not an isometry. □

In a completely analogous way, one can prove the following result.

Theorem 3.8. *For each $n \in \mathbf{N}$ let (X_n, d_n) be a metric space with d_n bounded by* 1. *Let (X, ρ) be the product $\prod_{n\in\mathbf{N}} X_n$ endowed with the metric*

$$\rho\big((x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}}\big) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n).$$

Then, if (X, ρ) is an EC-space, each factor must be an EC-space.

We have shown that the completion of an NEC-space need not be an NEC-space. It is also true that the completion of an EC-space need not be an EC-space, as the next theorem will show.

Theorem 3.9. *The space* $(\mathbf{R}-\mathbf{Z}, |\cdot|)$ *is an EC-space.*

Proof. Let f be a nonexpansive bijection from $\mathbf{R}-\mathbf{Z}$ onto itself. Then f is continuous, and since f is injective, it must map each component $(n, n+1)$ of $\mathbf{R}-\mathbf{Z}$ onto an open interval. Since an open interval cannot be expressed as the union of two or more disjoint open intervals and f is bijective, it follows that for each $n \in \mathbf{Z}$ there exists $n^* \in \mathbf{Z}$ such that $f((n, n+1)) = (n^*, n^*+1)$.

For a fixed $n \in \mathbf{Z}$ consider the mapping $g : (n, n+1) \to (n, n+1)$ defined by $g(x) = f(x) - n^* + n$. Since f is nonexpansive, g is also, and because $(n, n+1)$ is an EC-space, g must be an isometry. Hence, for each $n \in \mathbf{Z}$, the restriction of f to $(n, n+1)$ is an isometry.

Now the only isometries from $(n, n+1)$ onto itself are $x \mapsto x$ and $x \mapsto 2n+1-x$, and both of these isometries have $n+1/2$ as a fixed point. It follows that $f(n+1/2) = n^*+1/2$ for each integer n. Since f restricted to $S = \{k+1/2\colon\ k \in \mathbf{Z}\}$ is a nonexpansive bijection onto S and S is an EC-space (because it is isometric to $\mathbf{Z}$), the restriction $f|_S$ is an isometry. Hence, there is an integer k such that either $f|_S(x) = k+x$ for all $x \in S$ or $f|_S(x) = k-x$ for all $x \in S$.

Suppose there is a $k \in \mathbf{Z}$ such that $f|_S(x) = k+x$ for all $x \in S$, and let $y \in \mathbf{R}-\mathbf{Z}$. Then there exists $z \in S$ such that such that $z \leqslant y < z+1$. Now

$$\bigl|k+z-f(y)\bigr| = \bigl|f(z)-f(y)\bigr| \leqslant |z-y| = y-z$$

and

$$\bigl|k+z+1-f(y)\bigr| = \bigl|f(z+1)-f(y)\bigr| \leqslant |z+1-y| = z+1-y.$$

These two inequalities imply that $f(y) = k+y$. Thus $f(x) = k+x$ for all $x \in \mathbf{R}-\mathbf{Z}$. In a similar manner it can be shown that if there is a $k \in \mathbf{Z}$ such that $f|_S(x) = k-x$ for all $x \in S$, then $f(x) = k-x$ for all $x \in \mathbf{R}-\mathbf{Z}$. In either case f is an isometry. □

Since $\overline{\mathbf{R}-\mathbf{Z}} = \mathbf{R}$, the completion of $\mathbf{R}-\mathbf{Z}$ is an NEC-space.

As we stated earlier, the intersection of two EC-spaces need not be an EC-space. The following example will demonstrate this. Let

$$T = \bigcup_{n=0}^{\infty}\left(-n-\frac{1}{2}, -n+\frac{1}{2}\right) \cup \bigcup_{n=1}^{\infty}\left(2n-\frac{1}{4}, 2n+\frac{1}{4}\right).$$

Then T can be shown to be an EC-space using an argument similar to that used in the previous theorem. Notice that here it is important to use intervals of two different lengths. If they were all of the same length, then the mapping that shifts each component interval to its neighbor to the right would be a nonisometric noncontraction. Now $\mathbf{Z}$ is also EC-plastic, but $\mathbf{Z} \cap T$ is an NEC-space.

We summarize the results of this section in Table 1.

Table 1

	Subspace	Dense subspace	Union	Cartesian intersection	Product
EC-space	–	–	–	–	–
NEC-space	–	–	–	–	+

4. Hereditarily EC-spaces

In this section we consider spaces that are hereditarily EC-plastic. It is easy to see that no metric space can be hereditarily NEC-plastic because every finite subspace is an EC-space. But hereditarily EC-spaces do exist. In fact any totally bounded metric space has this property.

As motivation for the next proposition, consider the following example. Let S be the subspace of $(\mathbf{R}, |\cdot|)$ defined by $S = \{2^n\colon n \in \mathbf{Z}\}$, and let $f : S \to S$ be defined by $f(x) = 2x$. Then f is a noncontractive bijection but not an isometry. (We can visualize the action of f on S by thinking of the points of S as beads on an elastic string, $[0, \infty)$ and f pulling each bead to the right as it stretches the string.) It follows that $\mathbf{R}$ is not hereditarily EC-plastic nor is any space containing an isometric copy of S.

Following [4] we say that for a metric space (X, d) if $d(x, y) + d(y, z) = d(x, z)$, then y is *between* x and z. If in addition $d(x, y) = d(y, z)$, then y is called a *midpoint* of x and z. The space (X, d) is called a *convex metric space* if each pair of points has at least one midpoint.

Proposition 4.1. *Let (X, d) be a convex metric space. If X is a hereditary EC-space, then it is bounded.*

Proof. Suppose X is unbounded. We will construct an NEC-subspace Y of X. Let x and x_0 be distinct points of X. Since X is unbounded, there is a point $x_1 \in X$ such that

$$d(x_1, x) > d(x, x_0) \quad \text{and} \quad d(x_1, x_0) > d(x, x_0).$$

Having chosen $x_1, \ldots, x_k$ inductively, we choose x_{k+1} such that

$$d(x_{k+1}, y) > \max\{d(x_m, x_n), d(x, x_n)\colon m, n = 0, \ldots, k\}$$

for all $y = x, x_0, \ldots, x_k$. By convexity, there is x_{-1} such that $d(x_{-1}, x) = (1/2)d(x, x_0)$. For each $k = 2, 3, \ldots$, let x_{-k} be a midpoint of x and x_{-k+1}. Finally let $Y = \{x, x_i\colon i \in \mathbf{Z}\}$ and let $f : Y \to Y$ be defined by

$$f(y) = \begin{cases} x, & \text{if } y = x; \\ x_{i+1}, & \text{if } y = x_i. \end{cases}$$

Then f is a noncontractive bijection, but it is not an isometry. □

Corollary 4.2. *Let X be a convex subset of the euclidean space $(\mathbf{R}^n, d)$. Then (X, d) is a hereditary EC-space if and only if X is bounded.*

Proof. If X is bounded, then it is totally bounded (as a subspace of $\mathbf{R}^n$). Hence, any subspace of X is totally bounded; thus, an EC-space. □

As it turns out, the condition that the metric space in Proposition 4.1 be convex is stronger than is needed. We need the existence of an accumulation point only.

Theorem 4.3. *Let (X, d) be an unbounded metric space with at least one accumulation point. Then X contains an NEC-space.*

Proof. Let x be an accumulation point of X, and let x_0 be another point of X. Let r be any positive real number such that $r < \sqrt{2} - 1$. Since X is unbounded, there exists for each integer n a point x_n such that $d(x_{n+1}, x) \leqslant r d(x_n, x)$. Consider the subspace $Y = \{x_n \colon n \in \mathbf{Z}\}$ and the bijection $f : Y \to Y$ defined by $f(x_n) = x_{n+1}$ for each $n \in \mathbf{Z}$. We will show that f is nonexpansive but not an isometry.

Let $m, n \in \mathbf{Z}$ with $n > m$. Then

$$\begin{aligned} d(x_{n+1}, x_{m+1}) &\leqslant d(x_{n+1}, x) + d(x, x_{m+1}) \leqslant r^{n-m+1} d(x_m, x) + r d(x, x_m) \\ &= (r^{n-m+1} + r) d(x, x_m). \end{aligned}$$

Also

$$d(x, x_m) \leqslant d(x, x_n) + d(x_n, x_m) \leqslant r^{n-m} d(x, x_m) + d(x_n, x_m);$$

so

$$(1 - r^{n-m}) d(x, x_m) \leqslant d(x_n, x_m).$$

It follows that

$$d\big(f(x_n), f(x_m)\big) = d(x_{n+1}, x_{m+1}) \leqslant \frac{r(r^{n-m} + 1)}{1 - r^{n-m}} d(x_n, x_m).$$

Now

$$\frac{r(r^{n-m} + 1)}{1 - r^{n-m}} \leqslant \frac{r(r+1)}{1-r} < 1,$$

because $0 < r < \sqrt{2} - 1$. Hence $d(f(x_n), f(x_m)) < d(x_n, x_m)$ for all $n, m \in \mathbf{Z}$ with $n \neq m$. □

As the following example will show, boundedness does not guarantee that a metric space will be a hereditary EC-space, or even an EC-space.

Example 4.4. Let S and T be disjoint infinite sets, and let d be defined on $X = S \cup T$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x, y \in S; \\ 2, & \text{otherwise.} \end{cases}$$

Then (X, d) is a metric space. Now let $s_0 \in S$ and $t_0 \in T$, and let $g : S - \{s_0\} \to S$ and $h : T \to T - \{t_0\}$ be bijections. Define $f : X \to X$ by

$$f(x) = \begin{cases} t_0, & \text{if } x = s_0; \\ g(x), & \text{if } x \in S - \{s_0\}; \\ h(x), & \text{if } x \in T. \end{cases}$$

It can easily be verified that f is a noncontractive bijection. However, f is not an isometry because

$$d\big(f(s_0), f(x)\big) = d\big(t_0, g(x)\big) = 2 > 1 = d(s_0, x)$$

for any $x \in S - \{s_0\}$. Hence X is not an EC-space.

5. Closing remarks

From the examples given previously, we see that an EC-space need not be compact, complete, or bounded. It is an open question whether there exists a simple characterization of these spaces. The same question remains open for CE-spaces. (See Knaster's question in [6].)

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