COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
23. TOPOLOGY, BUDAPEST (HUNGARY), 1978.

SEPARATE ALMOST CONTINUITY AND JOINT CONTINUITY

Z. PIOTROWSKI

R. Baire [2] was one of the first mathematicians who considered separate continuity vis-à-vis joint continuity. The general problem (P), explicitly formulated by I. Namioka in [16], is the following: find conditions on topological spaces X, Y and Z so that each separately continuous function $f\colon X \times Y \to Z$ (i.e., function continuous in each variable while the other variable is fixed) is jointly continuous at points of a "substantial" (in some topological sense) subset of $X \times Y$. Namioka answered this problem in the case where X is strongly countably complete, regular, Y is locally compact and σ-compact (he deduced the theorem from the special case where Y is compact) and Z is pseudometrizable, and the "substantial" set $S = A \times Y$, where A is a dense G_δ in X. (He asked, if this theorem remains valid, when X is only assumed to be Baire (see [16], Remarks 1.3 (b), p. 520). I do not know, whether this question is settled). The literature concerning separate continuity vis-à-vis joint continuity is large, for some results in this direction (see [1]-[4], [8]-[10], [12], [14], [15], [19]-[23].

There are many applications of this subject such as Ellis' theorem [6], on separately continuous actions of locally compact groups on locally compact spaces and the existence of denting points on weakly compact convex subsets of locally convex metrizable linear topological spaces. For other applications, see [1] and [16].

The purpose of this paper is to give an answer to problem (P). Recall that a space X will be called *quasi-regular* if for every nonempty, open subset U of X, there is a nonempty, open set V such that $\bar{V} \subset U$ (see [14], p. 39 and [18], p. 164). A function $f\colon X \to Y$ is called *quasi-continuous at a point* $x \in X$, if for arbitrary open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$, we have $A \cap \operatorname{Int} f^{-1}(H) \neq \phi$. A function $f\colon X \to Y$ is called *quasi-continuous*, if it is quasi-continuous at each point x of X ([14], p. 39, compare [11], p. 186). A function $f\colon X \times Y \to Z$ (X, Y, Z arbitrary topological spaces) is said to be *quasi-continuous at* $(p, q) \in X \times Y$ *with respect to the variable* y, (see [19], Definition 3), if for every neighbourhood N of $f(p, q)$ and for every neighbourhood $U \times V$ of (p, q), there exists a neighbourhood V' of q, with $V' \subset V$, and a nonempty open $U' \subset U$, such that for all $(x, y) \in$ $\in U' \times V'$ we have $f(x, y) \in N$. If f is quasi-continuous with respect to the variable y at each point of its domain, it will be called *quasi-continuous with respect to* y. The definition of a function f that is quasi-continuous with respect to x is quite similar. If f is quasi-continuous with respect to x and y, we will say that f is *symmetrically quasi-continuous*.

– An immediate consequence from Theorem 2 of [19] is the following

Theorem 1. *Let* X *and* Y *be locally countably compact, quasi-regular spaces and* Z *be a metric one. If a function* $f\colon X \times Y \to Z$ *is symmetrically quasi-continuous, then the set of points of joint continuity of* f *contains a dense subset of* $X \times \{y\}$ *and* $\{x\} \times X$, *for all* $x \in X$ *and* $y \in Y$.

The available answers to problem (P), except [16] and [22] (however in [22], the separately continuous function f is assumed to be *bounded*), require that either X or Y be metrizable or satisfy some sort of countability condition. Following Namioka [16], p. 515, this requirement severely restricts their applications.

We note here that, in Theorem 1, the spaces X or Y need not be metrizable and need not satisfy any countability condition, however f is assumed to be symmetrically quasi-continuous, instead of being separately continuous. The relations between these two generalized continuity notions were considered in [19].

We give here without (standard) proof the following

Lemma 2. *A first countable, locally countably compact space is regular.*

Lemma 3 ([19], Lemma 1). *Every quasi-regular, locally countably compact space is Baire.*

Lemma 4 ([19], Corollary 1). *Let X and Y be first countable, Baire spaces and Z be a regular one. If a function $f: X \times Y \to Z$ is separately continuous, then f is symmetrically quasi-continuous.*

From Lemmas 2-4 and Theorem 1 we deduce the following

Theorem 5. *Let X and Y be first countable, locally countably compact and Z be metric. If a function $f: X \times Y \to Z$ is separately continuous, then the set of points of joint continuity of f is dense in $X \times \{y\}$ and $\{x\} \times Y$, for all $x \in X$ and $y \in Y$.*

We get the following Theorem 6 from Theorem 2 of [17] and Propoziţia 2 of [5], p. 985.

Theorem 6. *Let X be Baire, Y be first countable, with $X \times Y$ being Baire and Z be separable metric (in fact, regularity and second countability is needed) and let a function $f: X \times Y \to Z$ be separately quasi-continuous, then there is a dense, G_δ set $S \subset X \times Y$ such that f is jointly continuous at each point of S.*

The following reduction of problem (P) is, maybe, of interest: Let R denote the real line. Determine possibly the largest class of topological spaces X and Y for which separate continuity of a function $f: X \times Y \to R$ implies its B measurability of class 1 (see in this direction e.g. [7], Theorem 3, p. 147). Now, by an analogon of Lebesgue – Hausdorff Theorem

([13], Theorem, p. 393). f would be analytic representable of class 1. Thus, there is a residual set $S \subset X \times Y$ such that f is jointly continuous at each point of S.

The author suspects, that another answer to problem (P) may be obtained using similar ideas to those of R. Kershner [12], where also a converse of problem (P) is solved(!), or H. Hahn [9] (§ 39, Theorem 39.3.6) is a good answer to problem (P) which has been overlooked in the past. Namely, Hahn's Theorem is an analogon of Namioka's result ([16], Theorem 1.2, p. 517), in the case when X and Y are assumed to be metric.

Professor I. Namioka has kindly informed the author that Dr. Michel Talagrand proved Theorem 2.1 of [16] under the following hypothesis: X is compact (X is Čech-complete is sufficient) and Y is a "special" K-analytic space. This has interesting applications in Banach space theory.

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Z. Piotrowski

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.