On Volterra Spaces III: Topological Operations*

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Abstract

Over 100 years ago Volterra showed if $f : \mathbb{R} \to \mathbb{R}$ is a function whose sets of points of continuity and discontinuity are both dense then there is no function $g : \mathbb{R} \to \mathbb{R}$ whose set of points of continuity is precisely the set of points of discontinuity of f. This led to the definition of Volterra and strongly Volterra spaces. In this paper the behaviour of these spaces under taking subspaces and superspaces, images and preimages and products is studied. These concepts are also tied to other concepts related to the notion of a non-empty open set not being expressible as the union of few members of an ideal of thin subsets of a space.

1 Introduction

The decades of the 1860's and 1870's were very fruitful for the development of Modern Analysis. In fact, the class of functions satisfying the Lipschitz condition was introduced in 1864, Riemann-integrable functions were already studied in 1867 and in 1870 H.Hankel [10] introduced pointwise discontinuous functions. Apparently, the latter class of functions became the main object of studies in real function theory until, at least, the appearance of the works of H.Lebesgue.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *pointwise discontinuous*, abbreviated PWD, if the set $C(f)$ of points of continuity of f is dense.

It can be shown, see, e.g.,[14, Theorem 7.4, p.33], that if $X = Y = \mathbb{R}$ then:

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- f is PWD if and only if the set $D(f)$ of points of discontinuity of f is of first category.

Clearly, the definition of pointwise discontinuity can be given for functions between topological spaces X and Y. Using the same arguments as ones used in the proof given in [14] of the statement above, one can show that it holds if X is a (topological) Baire space and Y is metric.

In 1881, Vito Volterra who was not yet 20 years old, studied properties of PWD real-valued functions of real variable. He proved the following result:

(*) Let $f : \mathbb{R} \to \mathbb{R}$ be a PWD function. Then there is no other PWD function $g : \mathbb{R} \to \mathbb{R}$, which is continuous at the points where f is discontinuous and discontinuous at the points where f is continuous.

Observe that (*) is equivalent to the following condition which will be labeled "Volterra's theorem" in the sequel [15]:
Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that both the set of points where f is continuous and its complement are dense. Then there is no function $g : \mathbb{R} \to \mathbb{R}$ such that the set of points where g is continuous is precisely the set of points where f is discontinuous.
Thus, for example, while there is a function whose set of points of continuity is precisely the irrationals, there is no function whose set of points of continuity is the rationals. Development of these ideas in previous work of one or more of the present authors, [6, 7, 8], led to the definitions of Volterra and strongly Volterra repeated in section 1 below. Related work on Baire spaces, which have a close connection with Volterra spaces, is found in [1, 11].

In this paper we continue the study of Volterra spaces, looking at their behaviour under taking subspaces and superspaces, images and preimages and products. Previously [8] we used the term "Volterra" to refer to a property which we considered to be closest to the statement of Volterra's theorem, and "strongly Volterra" to refer to a related but strictly stronger property. In this paper we change this terminology to reserve the title "Volterra" for the global property (previously "strongly Volterra") and what we previously called "Volterra" we here call "weakly Volterra". This is the same terminology as used in [9]. We obtain a new criterion (Corollary 4.2) for a space to be Volterra which parallels the criterion for a space to be Baire in terms of the second category of its open subspaces. In the final section we connect this and related concepts to the notion of a non-empty open subspace not being able to be expressed as the union of few members of an ideal or σ-ideal of thin subsets of the space.

2 Definitions

For any function $f : X \to Y$ we denote by $C(f)$ the set of points at which f is continuous. Throughout we assume that our topological spaces are non-empty.

Definition 2.1 [8] *A topological space X is* weakly Volterra *if for each pair $f, g : X \to \mathbb{R}$ of functions such that $C(f)$ and $C(g)$ are both dense in X we have $C(f) \cap C(g) \neq \varnothing$.*

Definition 2.2 [8] *A topological space X is* Volterra *if for each pair $f, g : X \to \mathbb{R}$ of functions such that $C(f)$ and $C(g)$ are both dense in X the set $C(f) \cap C(g)$ is dense in X.*

The following equivalent conditions are shown in [8].

Proposition 2.3 *For any non-empty topological space the following are equivalent:*

1. *X is weakly Volterra;*
2. *for each pair A, B of dense G_δ subsets of X we have $A \cap B \neq \varnothing$;*
3. *for each pair $C, D \subset X$ of F_σ subsets such that $C \cup D = X$, either $\mathring{C}$ or $\mathring{D}$ is non-empty;*
4. *for each pair Y, Z of developable spaces and each pair $f : X \to Y$ and $g : X \to Z$ of functions for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is non-empty.*

Recall that a subset $S \subset X$ is *boundary* or *codense* if its complement in X is dense.

Proposition 2.4 *For any topological space the following are equivalent:*

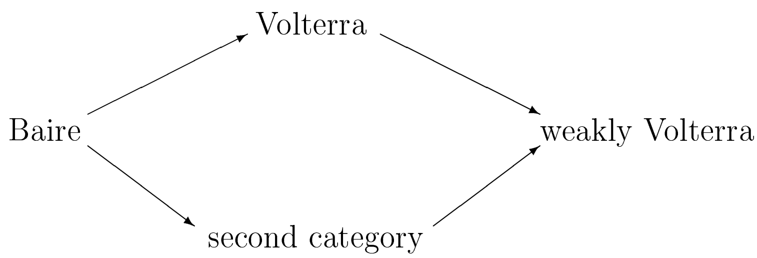
1. *X is Volterra;*
2. *for each pair A, B of dense G_δ subsets of X the set $A \cap B$ is dense;*
3. *for each pair $C, D \subset X$ of boundary F_σ subsets of X the set $C \cup D$ is boundary;*
4. *for each pair Y, Z of developable spaces and each pair $f : X \to Y$ and $g : X \to Z$ of functions for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is dense.*

Recall also the following definition.

Definition 2.5 *A topological space is* Baire *if the intersection of a sequence of open dense subsets is dense.*

Remarks. Gruenhage and Lutzer ([9]) have shown that a space that is T_3 and has a dense metric subspace, or is T_3, metacompact, first countable and is a σ-space, is Volterra if and only if it is Baire.

As noted in [8] we have the following relationship between the four properties Baire, second category, Volterra and weakly Volterra, with no other relationships between individual pairs of the properties.



3 Examples

In this section we present a number of examples which illustrate some of the behaviour of Volterra and weakly Volterra spaces.

Example 3.1 *The discrete and indiscrete spaces.*

As Baire spaces, both of these spaces are Volterra. Thus, as there are examples of spaces which are not weakly Volterra, it follows that reducing or enlarging the topology may destroy the (weak) Volterra property.

Example 3.2 *The rational numbers.*

The space $\mathbb{Q}$ is not weakly Volterra. Indeed, letting

$$\mathbb{Q}_o = \{\tfrac{p}{q} \mid q \text{ is odd}\}, \quad \mathbb{Q}_e = \{\tfrac{p}{q} \mid q \text{ is even}\},$$

when $\frac{p}{q}$ are in their lowest terms, we see that $\mathbb{Q}_o$ and $\mathbb{Q}_e$ are disjoint, dense G_δ subsets of $\mathbb{Q}$.

Example 3.3 *A space which is not weakly Volterra but has disjoint dense G_δ-subspaces which are Volterra.*

Let $X = \mathbb{N}$. Let O denote the odd positive integers and E the even positive integers. For each pair (m, n) of positive integers let

$$U_{m,n} = \{x \in X \mid \text{either } x \geq 2m - 1 \text{ and } x \in O \text{ or } x \geq 2n \text{ and } x \in E\}.$$

Then $\{\varnothing\} \cup \{U_{m,n} \mid m, n \in \mathbb{N}\}$ is a topology on X. It is readily checked that the two subsets O and E are dense G_δ-subsets and as subspaces are Volterra. From $O \cap E = \varnothing$ it follows that X is not weakly Volterra.

Example 3.4 *A space which is Volterra and has disjoint dense subspaces which are Volterra.*

Let $(X, \mathcal{T})$ be a separable, completely metrisable space with no isolated points. As completely metrizable, it is Baire, thus Volterra. It is known [13, p.514], see also [11, Theorem 2.6, p.21], that there is a subset $Z \subset X$ such that: both Z and $X - Z$ are dense in X, have cardinality continuum and are Baire spaces in the strong sense (= every nonempty closed subspace is of second category in itself). To get such a subset Z just use the following construction due to Bernstein:the cardinalities of X and $\mathcal{T}$ are both the continuum. Thus, each G_δ subset of X without isolated points has cardinality the continuum and there are exactly continuum many of them. Then construct Z by well-ordering X and the collection of G_δ sets and carefully remove points from G_δ sets. Thus both Z and $X - Z$ are Baire spaces, see [11, Proposition 2.1, p.18], and comments above. Thus both the sets are Volterra. It is interesting that the sets Z and $X - Z$ cannot be completely metrisable, since a Hausdorff space X cannot be the union of two or more pairwise disjoint, topologically complete, dense subspaces, [16].

Example 3.5 *A space which is not of second category, hence not Baire, but is Volterra.*

Let $X = [0, \infty)$ with topology having basis

$$\{[a, \infty) - F \mid a \in X \text{ and } F \text{ is a finite subset of } X\}.$$

Example 3.6 *A space which is not Volterra, hence not Baire, but is of second category, hence weakly Volterra.*

Let $X = \mathbb{R}_- \cup \mathbb{Q}_+$ with topology inherited from the reals, where $\mathbb{R}_-$ denotes the non-positive reals and $\mathbb{Q}_+$ the non-negative rationals.

Example 3.7 [5] *Another space which is weakly Volterra but not Volterra.*

Retopologise the plane by taking as subbasis the usual topology together with the set $\mathbb{Q} \times \{0\}$. This space is weakly Volterra because its open subspace consisting of the plane less the x-axis is Baire. However the two sets

$$\{(x,y) \mid \text{either } x \in \mathbb{Q}_o \text{ or } y \neq 0\ \} \text{ and } \{(x,y) \mid \text{either } x \in \mathbb{Q}_e \text{ or } y \neq 0\ \},$$

where $\mathbb{Q}_o$ and $\mathbb{Q}_e$ are as in Example 3.2, are dense G_δ subsets whose intersection is not dense.

Example 3.8 *The countable cloud space.*

Let $X = Y \cup Z$ be the subspace of the plane where

$$Y = \{(x,0) \mid x \in \mathbb{Q}\}, \quad Z = \{(\tfrac{p}{q}, \tfrac{1}{q}) \mid \tfrac{p}{q} \in \mathbb{Q}\}.$$

The countable cloud space appears in [1] as an example of a metric Baire space. Therefore, it is Volterra. Notice that the countable cloud space contains a closed subspace which is not weakly Volterra, viz the rationals in the x-axis.

Example 3.9 [4] *A metric Baire space whose square is not even weakly Volterra.*

This example is described in [3, 4]. For any function $f : \omega \to \omega_1$ set $f^* = sup\{f(n) \mid n \in \omega\}$. For any stationary subset $A \subset \omega_1$ define

$$\hat{A} = \{f : \omega \to \omega_1 \mid f^* \in A\}.$$

$\hat{A}$ is topologised by the metric ρ given by $\rho(f,g) = 2^{-n}$ whenever $f, g \in \hat{A}$, where n is the least integer so that $f|n \neq g|n$. Then by [3, 4] $\hat{A}$ is Baire. Now suppose that $A_1, A_2 \subset \omega_1$ are two disjoint stationary subsets: then $\hat{A}_1 \times \hat{A}_2$ is not even weakly Volterra. Indeed, for each $n \in \omega$ let

$$U_n = \{(f_1, f_2) \in \hat{A}_1 \times \hat{A}_2 \mid f_1(n) < f_2^* \text{ and } f_2(n) < f_1^*\}.$$

Each set U_n is open in $\hat{A}_1 \times \hat{A}_2$. Thus the set $U_e = \cap_{n\in\omega} U_{2n}$ is G_δ. U_e is also dense in $\hat{A}_1 \times \hat{A}_2$. To verify this, suppose given $(f_1, f_2) \in \hat{A}_1 \times \hat{A}_2$ and a positive integer m: we must show that $V \cap U_e \neq \varnothing$, where

$$V = \{(g_1, g_2) \in \hat{A}_1 \times \hat{A}_2 \mid g_i|m = f_i|m\}.$$

Choose $a_i \in A_i$ so that for each $n \leq m$ we have $f_i(n) < a_1$ and $f_i(n) < a_2$. Define g_i as follows:

$$g_i(n) = \begin{cases} f_i(n) & : \quad \text{if } n \leq m \\ f_i(1) & : \quad \text{if } n > m \text{ and } n \text{ is even} \\ a_i & : \quad \text{if } n > m \text{ and } n \text{ is odd.} \end{cases}$$

Then $g_i^* = a_i$, so when n is even we have $g_1(n) < g_2^*$ and $g_2(n) < g_1^*$; thus $(g_1, g_2) \in U_e$. Note also that $g_i|m = f_i|m$ so $(g_1, g_2) \in V$. Thus $V \cap U_e \neq \varnothing$, so U_e is dense in $\hat{A}_1 \times \hat{A}_2$. Similarly $U_o = \cap_{n\in\omega} U_{2n+1}$ is a dense G_δ subset of $\hat{A}_1 \times \hat{A}_2$. However $U_e \cap U_o = \varnothing$ because if we had $(f_1, f_2) \in U_e \cap U_o$ then for all $n \in \omega$ we would have $f_1(n) < f_2^*$ so that $f_1^* \leq f_2^*$ so by symmetry $f_1^* = f_2^*$, contrary to $A_1 \cap A_2 = \varnothing$. We conclude from Proposition 2.3(2)that $\hat{A}_1 \times \hat{A}_2$ is not weakly Volterra.

4 Preservation under Subspaces, Superspaces and Products

Example 3.2 shows that the property of being (weakly) Volterra need not be inherited by arbitrary subspaces. In this section we show that for some classes of subspace these properties are inherited. This leads to a new characterisation of when a space is Volterra. We also find circumstances in which a union of spaces which are (weakly) Volterra is also (weakly) Volterra.

Theorem 4.1 *Suppose that X is a Volterra space and $S \subset X$ a subset containing a G_δ-subset G such that int$\bar{G}$ is dense in S. Then S is a Volterra space. Furthermore, if X is merely assumed to be weakly Volterra and $S \subset X$ is a dense G_δ-subset then S is also weakly Volterra.*

Proof. Since int$\bar{G}$ is dense in S we have $\bar{S} = \overline{S \cap \text{int}\bar{G}}$, so that $\bar{G} \subset \bar{S} \subset \overline{S \cap \text{int}\bar{G}} \subset \bar{G}$, hence $\bar{G} = \bar{S}$.

Let A and B be G_δ sets in X which are dense in S (i.e. $\overline{S \cap A} = \bar{S} = \overline{S \cap B}$). Define $G' = G \cup (X - \bar{S})$, $A' = A \cup (X - \bar{S})$ and $B' = B \cup (X - \bar{S})$. Then G', A' and B' are dense G_δ sets in X and hence so is $G' \cap A' \cap B'$ as X is Volterra. From $\bar{G} = \bar{S}$ it follows that int$\bar{G} \subset \overline{G \cap A \cap B}$. Therefore

$$\bar{S} \subset \overline{S \cap \text{int}\bar{G}} \subset \overline{\text{int}\bar{G}} \subset \overline{G \cap A \cap B} \subset \overline{S \cap A \cap B},$$

so that $A \cap B$ is dense in S.

Finally if X is weakly Volterra and $S \subset X$ is a dense G_δ-subset then the conclusion follows from Proposition 2.3(2). ∎

Note that S satisfies the hypothesis of Theorem 4.1 if S is an open or a regular closed or a dense G_δ-subset of X.

In Theorem 4.1 we cannot extend the hypothesis concerning S to just “closed”. Indeed, the cloud space of Example 3.8 shows that a closed subspace of a Volterra space need not even be weakly Volterra: the space X is Volterra but its closed subspace Y is not weakly Volterra, as noted in Example 3.2.

It is not the case that an arbitrary open or regular closed subspace of a weakly Volterra space need be Volterra, so the weakly Volterra analogue of Theorem 4.1 does not hold. Indeed, we may take $X = \mathbb{R}_- \cup \mathbb{Q}_+$ as in Example 3.6 and $S = \mathbb{Q}_+ \cap (0, \infty)$ (respectively $\mathbb{Q}_+$). Then S is an open (respectively regular closed) subspace of the weakly Volterra space X, but, as noted in Example 3.2, S is not weakly Volterra.

The following corollary of Theorem 4.1 emphasises how the relationship between weakly Volterra and Volterra parallels that between second category and Baire. Recall that a space is Baire if and only if each non-empty open subspace is of second category.

Corollary 4.2 *The space X is Volterra if and only if each non-empty open subspace is weakly Volterra.*

Proof. By Theorem 4.1 we need only show that if each non-empty open subspace is weakly Volterra then X is Volterra. Suppose that A and B are two dense G_δ-subsets of X. Let U be a non-empty open subset of X. Then $U \cap A$ and $U \cap B$ are two G_δ-subsets of U which are dense in U. It follows that $U \cap A \cap B = (U \cap A) \cap (U \cap B) \neq \varnothing$ as U is weakly Volterra. Thus $A \cap B$ is dense in X so X is Volterra. ∎

We now consider whether the property of being (weakly) Volterra is preserved when we pass to superspaces. Note from Example 3.3 that we may have a space E which is Volterra yet is a dense G_δ-subspace of a space X which is not even weakly Volterra.

Proposition 4.3 *Suppose that $\mathcal{U}$ is a collection of open subsets of the space X whose union is dense in X. Then:*

1. *if there is some non-empty $U \in \mathcal{U}$ such that U is weakly Volterra then X is weakly Volterra;*

2. *if each member of $\mathcal{U}$ is Volterra then X is Volterra.*

Proof. Because each member of $\mathcal{U}$ is open, it follows that for any dense subset D of X, the set $D \cap U$ is dense in U whenever $U \in \mathcal{U}$. Suppose that A and B are two dense G_δ-subsets of X. Let V be either the whole space X in case 1 or any non-empty open subset of X in case 2: it suffices to show that $V \cap A \cap B \neq \varnothing$. Choose $U \in \mathcal{U}$ such that $U \cap V \neq \varnothing$. Then $U \cap V$ is all of U in case 1 and is a non-empty open subset of U in case 2. By the observation at the beginning of the proof we have that $U \cap A \cap B$ are both dense in U. Thus by hypotheses in either case 1 or case 2 we have $U \cap V \cap A \cap B \neq \varnothing$, from which the results follow. ∎

Corollary 4.4 *The topological sum of a family of (weakly) Volterra spaces is (weakly) Volterra.*

Example 3.9 shows that in general the product of Baire spaces need not even be weakly Volterra.

5 Preservation under Images and Preimages of Functions

Let X be any space which is not weakly Volterra. As noted in Example 3.1, when X is retopologised with either the discrete or indiscrete topology, X becomes Volterra. Thus the identity function from X with the discrete (respectively indiscrete) topology to X with the original topology is a continuous (respectively open) function from a Volterra space to a space which is not weakly Volterra. It follows that the image of a Volterra space under either a continuous function or an open function need not even be weakly Volterra. Note that Example 3.7 also gives us an open function (viz the identity) from the (Volterra) plane to a space which is not Volterra.

Compare the following proposition with [5], theorem 1. We recall that a function $f : X \to Y$ is *feebly open* if for each set $A \subset X$ having non-empty interior we have $\text{int } f(A) \neq \varnothing$.

Proposition 5.1 *Suppose that $f : X \to Y$ is continuous and feebly open. If X is weakly Volterra then so is Y. If X is Volterra and f is surjective then Y is Volterra.*

Proof. Let A and B be two dense G_δ-subsets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are G_δ-subsets of X. Furthermore, if $U \subset X$ is non-empty and open then so is $\text{int} f(U) \subset Y$, so that $\text{int} f(U) \cap A \neq \varnothing$, and hence $U \cap f^{-1}(A) \neq \varnothing$. Thus $f^{-1}(A)$ is dense. Similarly $f^{-1}(B)$ is dense.

Let $V \subset Y$ be a non-empty open set, with $V = Y$ in the case where X is merely weakly Volterra: it remains to show that $V \cap A \cap B \neq \varnothing$. Now $f^{-1}(V) \subset X$ is non-empty and open and is all of X when $V = Y$, so in either case $f^{-1}(V) \cap f^{-1}(A) \cap f^{-1}(B) \neq \varnothing$, say $x \in f^{-1}(V) \cap f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in V \cap A \cap B$ as required. ∎

Remark: Recall that in case of Baire spaces, in order for f to preserve this property, f needs to be quasi-continuous (ie for each open $V \subset Y$ we have $f^{-1}(V) \subset \text{cl int} f^{-1}(V)$, [12]) and feebly open [5]. In our proof of the preservation of Volterra (resp. weakly Volterra) spaces we need the full strength of continuity and feeble openness. There are two places where continuity of f is used in the proof above: in deducing that $f^{-1}(A)$ and $f^{-1}(B)$ are open, and $f^{-1}(V)$ is open. We require only that $\text{int}f^{-1}(V) \neq \varnothing$ and this is guaranteed by quasi-continuity. However quasi-continuity does not suffice in the other application as if $A = \cap_{n\in\omega} A_n$, where A_n is open, then $U \cap (\cap_{n\in\omega} f^{-1}(A_n)) \neq \varnothing$ implies that for each n we have $U \cap \text{int}f^{-1}(A_n) \neq \varnothing$, say $x_n \in U \cap \text{int}f^{-1}(A_n)$, but this is not enough to ensure that $U \cap f^{-1}(A) \neq \varnothing$ as x_n may vary too much with n. Indeed, let X, E and O be as in Example 3.3 and define $f : E \to X$ by $f(e) = \frac{e}{2}$. Then f is an open function because if $U_{m,n} \cap E$ is open in E then $f(U_{m,n} \cap E) = U_{\frac{n}{2}+1,\frac{n}{2}}$ if $n \in E$ and $f(U_{m,n} \cap E) = U_{\frac{n+1}{2},\frac{n+1}{2}}$ if $n \in O$. Also, f is quasi-continuous because for any open $U_{m,n} \subset X$, let k =max$\{2n, 2m-1\}$. Then $U_{1,k} \cap E \subset f^{-1}(U_{m,n})$ and since $U_{1,k} \cap E$ is dense in E it follows that $\text{int}f^{-1}(U_{m,n})$ is dense in $f^{-1}(U_{m,n})$.

Let $\hat{A}_1$ and $\hat{A}_2$ be as in Example 3.9, and let $\pi : \hat{A}_1 \times \hat{A}_2 \to \hat{A}_1$ be projection onto the first coordinate. Properties exhibited by $\hat{A}_1 \times \hat{A}_2$, $\hat{A}_1$ and π confirm that the pre-image of a Volterra space under an open, continuous function need not even be weakly Volterra even if the spaces are metric and the function has Volterra fibres.

Let $X = [0, 2] \cap \mathbb{Q}$ and $Y = \{0, 1\}$ and define $f : X \to Y$ by $f(x) = 0$ if $x \leq \sqrt{2}$ and $f(x) = 1$ if $x \geq \sqrt{2}$. Then f is continuous, open and closed but Y is Volterra whereas X is not weakly Volterra; again both spaces are metric.

Let $X = ([-1, 0] \cap \mathbb{Q}) \cup (\cup_{n\in\mathbb{N}}[2n-1, 2n])$, $Y = \mathbb{N}$ and define $f : X \to Y$ as follows: write $[-1, 0] \cap \mathbb{Q} = \{x_n \mid n \in \mathbb{N}\}$ and set $f(x) = n$ if either $x = x_n$ or $x \in [2n-1, 2n]$. X is second countable, Y is Volterra and f is open, closed and feebly continuous with compact fibres. However X is not Volterra (although it is weakly Volterra).

6 Generalisations

In this section we look at the concept of the possibility or otherwise of expressing a space as a union of few subsets which in some sense are thin. Thus a non-empty open subset of a Baire space cannot be expressed as the countable union of nowhere dense sets and Volterra spaces cannot be expressed as a finite union of boundary F_σ sets; in one case it is the nowhere dense sets which are 'thin' and in the other it is the boundary F_σ sets. Recall that for a set X, a family $\mathcal{I}$ of subsets is an *ideal* if for each $A \in \mathcal{I}$ and each $B \subset A$ we have $B \in \mathcal{I}$ and for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, and is a *σ-ideal* if in addition it is closed under countable unions.

As examples of (σ)-ideals in a topological space we have the following:

- $\mathcal{I}$ is the ideal generated by $\{ S \subset X \mid S \text{ is a boundary } \mathrm{F}_\sigma \text{ set} \}$;
- $\mathcal{J}$ is the σ-ideal generated by $\{ S \subset X \mid S \text{ is a boundary } \mathrm{F}_\sigma \text{ set} \}$;
- $\mathcal{K}$ is the σ-ideal generated by $\{ S \subset X \mid S \text{ is nowhere dense} \}$;
- $\mathcal{C}$ is the σ-ideal $\{ S \subset X \mid S \text{ is countable} \}$.

There are possibilities in other areas, such as the family of subsets of measure 0 of a measure space. Note that $\mathcal{I} \subset \mathcal{J} \subset \mathcal{K}$ and that if the space X is T_1 and has no isolated points then $\mathcal{C} \subset \mathcal{K}$.

The proof of the Proposition 6.1 below is a straightforward application of Proposition 2.4(3).

Proposition 6.1 *Let $\mathcal{I}$ be the ideal above. Then X is Volterra if and only if no non-empty open subset of X is in $\mathcal{I}$.*

Proposition 6.2 *Let $\mathcal{J}$ and $\mathcal{K}$ be the ideals above. Then the following are equivalent:*

(a) X is Baire;

(b) $\mathcal{K}$ contains no non-empty open subset of X;

(c) $\mathcal{J}$ contains no non-empty open subset of X.

Proof. (a)$\Rightarrow$(b): This is a straightforward application of one of the characterisations of when a space is Baire.

(b)$\Rightarrow$(c): Trivial.

(c)$\Rightarrow$(a): If X is not Baire then there exist open dense subsets U_n of X and a non-empty open subset U of X such that $U \cap (\cap_n U_n) = \varnothing$. For every n let $F_n = X - U_n$, and let $S = \cup_n F_n$. Each F_n is a boundary F_σ-set, so that $S \in \mathcal{J}$. However $U \subset S$, and hence $U \in \mathcal{J}$. ■

It is an immediate consequence of Propositions 6.1 and 6.2 together with the inclusion $\mathcal{I} \subset \mathcal{K}$ that every Baire space is Volterra. Indeed this inclusion is the essence of the relationship between the two concepts. This raises a number of interesting questions. For example we may ask whether the class of spaces X which satisfy the condition that no non-empty open subset is in $\mathcal{J}$ is of independent interest. Note that the weakly Baire spaces of [2] are precisely those T_1 spaces having no isolated points for which no non-empty open subset is in $\mathcal{C}$, and the inclusion $\mathcal{C} \subset \mathcal{K}$ captures the essence of the relationship between Baire and weakly Baire spaces.

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Subspace Table

	open	closed	regular closed	dense	dense G_δ
Weakly Volterra	No Ex 3.6	No Ex 3.6	No Ex 3.6	No Ex 3.2	Yes Thm 4.1
Volterra	Yes Thm 4.1	No Ex 3.8	Yes Thm 4.1	No Ex 3.2	Yes Thm 4.1

Superspace Table

	dense	dense G_δ	open cover
Weakly Volterra	No Ex 3.3	No Ex 3.3	Yes Prop 4.3
Volterra	No Ex 3.3	No Ex 3.3	Yes Prop 4.3

Function Image Table

	continuous	open	continuous, open	closed, perfect
Weakly Volterra	No Ex 3.1	No Ex 3.1	Yes Prop 5.1	No Ex 3.9
Volterra	No Ex 3.1	No Ex 3.1	Yes Prop 5.1	No Ex 3.9