1. Blumberg theorem.

In 1922 H. Blumberg [A1] proved that if \( X = Y = \mathbb{R} \), the reals then:

(*) for every \( f: X \to Y \), there exists \( D \subset X \), \( D \) dense in \( X \) such that \( f[D] \) is continuous.

Even for functions \( f: \mathbb{R} \to \mathbb{R} \), the set \( D \) in (*) cannot be made to have cardinality \( c \), see [A2].

(S. Baldwin [2] in [A5] showed recently that it is consistent with the axioms of set theory that the set \( D \) in (*) can always be chosen to be uncountably dense); the set \( D \) cannot be necessarily chosen so that for \( f: \mathbb{R} \to \mathbb{R} \) there exists a set \( N \subset \mathbb{R} \) of \( n \) - dense in \( \mathbb{R} \) such that \( f[N] \) is pointwise discontinuous (rel. \( \mathbb{N} \)).

2. Blumberg spaces.

Let \( Y = \mathbb{R} \). Call a space \( X \) Blumberg if (*) holds for \( Y = \mathbb{R} \). J.C. Bradford and C. Goffman [A3] proved that a metric space \( X \) is Blumberg if and only if \( X \) is Baire, i.e., a space in which open, nonempty subsets are of second category. The key lemma in their proof is Banach category theorem.

H.E. White Jr. [A4] extended Bradford-Goffman result to topological spaces \( X \) which have \( \sigma \)-disjoint pseudobases. He also showed that the reals \( R \) with the density topology is a Baire space not being Blumberg. W.A.R. Weiss in [9] of [A5] gave an example of a compact Hausdorff space which is not Blumberg.

3. The dynamics of Blumberg spaces.

It is easy to see that a dense, or a closed subspace of a Blumberg space need not be Blumberg. The Stone-Cech compactification of a dense subspace of a completely regular Blumberg space is a Blumberg space, a result due to R. Levy and R.H. McDowell [4] in [A6].

The Cartesian product of Blumberg spaces need not to be a Blumberg space, since there is a metric Baire space (hence Blumberg) whose square is not Baire. On the other hand, S. Todorcevic [A9] showed that there is a first countable space \( X \), which is not Blumberg, whereas \( X \times X \) is a Blumberg space. It follows from the above theorem that the image of a Blumberg space under an open and continuous function need not be Blumberg.

Consider the union \( X \) of the graph \( Z \) of the Riemann function, restricted to the rationals and the copy \( Y \) of the rationals in the \( N \)-axis. The natural projection of \( X \) onto \( Y \) (which is constant on \( Y \)) shows that even perfect, continuous functions do not necessarily preserve Blumberg spaces. In contrast, [A6] Blumberg spaces are preserved in preimages under irreducible surjections.

4. Varieties.

M. Valdivia [A10] showed that (*) holds for linear transformations, where \( X \) and \( Y \) are metrizable linear spaces, and \( X \) is of the second category.

L. Drewnowski [A11] proved that "dense subset" in Valdivia's theorem cannot be replaced with "dense linear subspace".

Also Blumberg sets - dense sets \( D \), appearing in (*) have been studied in connection with the characterizations of some almost continuous functions, e.g., quasicontinuous see [A12], section 6.

References


