TWO VARIATIONS OF THE CHOQUET GAME*

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Abstract. The Banach-Mazur game and the Choquet game are revisited to deduce new characterizations of sieve (almost) complete spaces. Some classical results of Choquet are extended to larger classes of spaces. By using the notion of sieve (almost) completeness, certain types of topological closed graph and open mapping theorems are established.

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1. Introduction

In [5], Choquet introduced two classes of topological spaces, namely α-favorable spaces and strongly α-favorable spaces, in terms of topological games. Recall that the *Banach-Mazur game* on a topological space X, denoted by $\mathrm{BM}(X)$, is a two-person infinite game. Two players, called β and α, alternatively choose non-empty open subsets of X with β starting first such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots$. Then α is said to *win* a play of $\mathrm{BM}(X)$ if $\bigcap_{n<\omega} U_n \neq \varnothing$. The space X is called *α-favorable* if α has a stationary winning strategy in $\mathrm{BM}(X)$. The other class of spaces is defined by a game called the *Choquet game*, denoted by $\mathrm{CH}(X)$. In $\mathrm{CH}(X)$, Player β chooses a point $x_0 \in X$ and its open neighborhood U_0. To respond to β's move, α chooses an open neighborhood V_0 of x_0 with $V_0 \subseteq U_0$. Then β responds to α by selecting a point $x_1 \in V_0$ and its open neighborhood U_1 with $U_1 \subseteq V_0$, and α chooses an open neighborhood V_1 of x_1 with $V_1 \subseteq U_1$, and so forth. Then a play

$$\mathbf{P} = \{(x_n, U_n, V_n) : x_n \in V_n \subseteq U_n \text{ and } U_{n+1} \subseteq V_n \text{ for all } n < \omega\}$$

of $\mathrm{CH}(X)$ is produced. We say that α *wins* the play $\mathbf{P}$ if $\bigcap_{n<\omega} U_n \neq \varnothing$. Furthermore, X is called *strongly α-favorable* if α has a stationary winning strategy in $\mathrm{CH}(X)$.

By a *strategy* for Player α, we mean a function defined for each legal finite sequence of moves of Player β; a strategy for Player β is defined similarly. A *stationary strategy* is a strategy which depends on the opponent's last move only. A *winning (stationary) strategy* for a player is a (stationary) strategy such that this player wins each play of the game no matter how the opponent

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moves according to the game. If α has a winning strategy in BM(X), then X is called *weakly α-favorable*. For every point $x \in X$, let $o(x)$ denote the collection of all open neighborhood of x, and $o(X) = \bigcup_{x\in X} o(x)$. Let σ be a strategy for α in CH(X). We call that a finite sequence $(\langle x_j, U_j\rangle)_{0\le j\le i}$ $(i \in \omega)$ or an infinite sequence $(\langle x_n, U_n\rangle)_{n<\omega}$ in $\bigcup_{x\in X}\{x\}\times o(x)$ a *σ-sequence* if

$$x_j \in \sigma(\langle x_0, U_0\rangle, ..., \langle x_j, U_j\rangle) \subseteq U_j \subseteq \sigma(\langle x_0, U_0\rangle, ..., \langle x_{j-1}, U_{j-1}\rangle)$$

for all $1 \le j \le i$, or

$$x_n \in \sigma(\langle x_0, U_0\rangle, ..., \langle x_n, U_n\rangle) \subseteq U_n \subseteq \sigma(\langle x_0, U_0\rangle, ..., \langle x_{n-1}, U_{n-1}\rangle)$$

for all $n \in \mathbb{N}$ respectively. Then σ is a winning strategy for α in CH(X) if $\bigcap_{n<\omega} U_n \neq \varnothing$ for every infinite σ-sequence $(\langle x_n, U_n\rangle)_{n<\omega}$ in $\bigcup_{x\in X}\{x\}\times o(x)$. If σ is a stationary strategy, or a (stationary) strategy for other games, then σ-sequences can be defined similarly. For more details and terminology on BM(X), CH(X) and other topological games, refer to [13].

It is well-known that (strong) α-favorability is closely related to topological completeness or Baireness. For instance, Choquet [5] has proved the following two results:

(†) A metrizable space is strongly α-favorable if and only if it is Čech complete.

(‡) A metrizable space is α-favorable if and only if it has a completely metrizable dense G_δ-subspace.

It is interesting to consider what will happen when the metrizability in the above results is reduced to some weaker topological properties. In this paper, we shall extend these two results by considering classes of spaces which are larger than that of metrizable spaces. To this end, we first study certain variations of the Choquet game. New characterizations of certain classes of spaces by using these variations are provided. In particular, sieve complete spaces, almost complete spaces, and monotonically p-spaces are considered. Then these classes of spaces shall be used to study topological versions of closed-graph and open mapping theorems.

2. MP-game and MC-game

Let X be a topological space. We shall consider two variations of CH(X), namely MP(X) and MC(X). In MP(X), the same players β and α and the same plays as those described in CH(X) except the winning rule. We say that α *wins* a play

$$\mathbf{P} = \{(x_n, B_n, A_n) : x_n \in A_n \subseteq B_n \text{ and } B_{n+1} \subseteq A_n \text{ for all } n < \omega\}$$

of MP(X) if either (i) $\bigcap_{n<\omega} B_n = \varnothing$; or (ii) $\bigcap_{n<\omega} B_n$ is a nonempty compact set such that for any open set W containing $\bigcap_{n<\omega} B_n$, there exists some $n < \omega$ with $B_n \subseteq W$. In MC(X), everything is the same as that in MP(X), except that only (ii) is satisfied for α to win a play of MC(X). A *sieve*

$(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ on X is a sequence of indexed covers $\{U_i : i \in I_n\}$ of X, together with maps $\pi_n : I_{n+1} \to I_n$ such that $U_i = X$ for all $i \in I_0$ and $U_i = \bigcup\{U_j : j \in \pi_n^{-1}(i)\}$ for all $i \in I_n$ and all $n < \omega$. Moreover, a *π-chain* for such a sieve is a sequence $(i_n)_{n<\omega}$ such that $i_n \in I_n$ and $\pi_n(i_{n+1}) = i_n$ for all $n < \omega$. A filterbase $\mathcal{F}$ on X is said to be *controlled* by a sequence $(U_n)_{n<\omega}$ of subsets of X if each U_n contains some $F \in \mathcal{F}$. Furthermore, if each filterbase controlled by $(U_n)_{n<\omega}$ clusters, then $(U_n)_{n<\omega}$ is called a *complete sequence* on X.

Theorem 2.1. *The following are equivalent for a regular space X.*

(a) *There is an open sieve $(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ on X such that for each π-chain $(i_n)_{n<\omega}$ with $\bigcap_{n<\omega} U_{i_n} \neq \varnothing$, $(U_{i_n})_{n<\omega}$ is a complete sequence.*

(b) *Player α has a stationary winning strategy in MP(X).*

(c) *Player α has a winning strategy in MP(X).*

Proof. (a) $\Rightarrow$ (b). Suppose that there exists an open sieve $(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ on X as described in (a). We inductively define a stationary strategy σ for α. First, suppose that Player β chooses his first move as $\langle x_0, B_0\rangle \in \bigcup_{x\in X}\{x\} \times o(x)$. Then Player α chooses an $i_0 \in I_0$ (where i_0 depends on x_0 only) such that $x_0 \in U_{i_0}$, and defines $\sigma(\langle x_0, B_0\rangle)$ to be any open neighborhood of x_0 such that $\overline{\sigma(\langle x_0, B_0\rangle)} \subseteq B_0 \cap U_{i_0}$. Player β responds to this move of α by selecting $\langle x_1, B_1\rangle \in \bigcup_{x\in X}\{x\} \times o(x)$ such that $x_1 \in B_1 \subseteq \sigma(\langle x_0, B_0\rangle)$. In return, α chooses an $i_1 \in \pi_0^{-1}(i_0)$ such that $x_1 \in U_{i_1}$, and defines $\sigma(\langle x_1, B_1\rangle)$ to be any open neighborhood of x_1 such that $\overline{\sigma(\langle x_1, B_1\rangle)} \subseteq B_1 \cap U_{i_1}$. Inductively, suppose that β has chosen a finite sequence $(\langle x_0, B_0\rangle, ..., \langle x_{n+1}, B_{n+1}\rangle)$ in $\bigcup_{x\in X}\{x\} \times o(x)$ such that $x_{j+1} \in B_{j+1} \subseteq \sigma(\langle x_j, B_j\rangle)$ for all $0 \leq j \leq n$, and α has chosen a finite sequence $(i_j)_{1\leq j\leq n}$ such that $i_j \in \pi_{j-1}^{-1}(i_{j-1})$ and $\overline{\sigma(\langle x_j, B_j\rangle)} \subseteq B_j \cap U_{i_j}$ for all $1 \leq j \leq n$. Next, Player α selects some $i_{n+1} \in \pi_n^{-1}(i_n)$ such that $x_{n+1} \in U_{i_{n+1}}$ (where i_{n+1} depends on x_{n+1} only), and defines $\sigma(\langle x_{n+1}, B_{n+1}\rangle)$ to be any open neighborhood of x_{n+1} such that $\overline{\sigma(\langle x_{n+1}, B_{n+1}\rangle)} \subseteq B_{n+1} \cap U_{i_{n+1}}$. This completes the definition of σ. To show that σ is a winning strategy for β, let $(\langle x_n, B_n\rangle)_{n<\omega}$ be a σ-sequence in MP(X). Then, by the above definition of σ, there exists a π-chain $(i_n)_{n<\omega}$ such that

$$x_n \in \sigma(\langle x_n, B_n\rangle) \subseteq \overline{\sigma(\langle x_n, B_n\rangle)} \subseteq B_n \cap U_{i_n}$$

for all $n < \omega$. If $\bigcap_{n<\omega} B_n = \varnothing$, there is nothing to prove. Otherwise, $\bigcap_{n<\omega} U_{i_n} \neq \varnothing$. To see that $(B_n)_{n<\omega}$ satisfies (ii) in MP(X), let $\mathcal{F}$ be a filterbase on $\bigcap_{n<\omega} B_n$. Then $\mathcal{F}$ is controlled by $(U_{i_n})_{n<\omega}$, and thus $\mathcal{F}$ has a cluster point. This implies that $\bigcap_{n<\omega} B_n$ is compact. Let W be an open set containing $\bigcap_{n<\omega} B_n$. Suppose that for every $n < \omega$ there exists a point $z_n \in B_n \setminus W$. For each $n < \omega$, we set $F_n = \{z_m : m \geq n\}$. Then $(F_n)_{n<\omega}$ is a filterbase controlled by $(U_{i_n})_{n<\omega}$, and thus it clusters at some point z. Clearly, z is also a cluster point of $(z_n)_{n<\omega}$ with $z \notin W$. But $z \in \bigcap_{n<\omega} B_n$. This is a contradiction. Thus, $B_n \subseteq W$ for some $n < \omega$.

(b) $\Rightarrow$ (c). It is trivial.

(c) $\Rightarrow$ (a). Suppose that α has a winning strategy σ in MP(X). Let $I_0 = \{i_0\}$ be an arbitrary singleton, and define $U_{i_0} = X$. For each $n > 0$, define I_n as the set of all finite σ-sequences of length $n+1$. Let $U_i = B_n$ for every $i = (\langle x_0, B_0\rangle, \cdots, \langle x_n, B_n\rangle) \in I_n$. Define $\pi_0 : I_1 \to I_0$ such that $\pi_0(i_1) = i_0$ for every $i_1 \in I_1$. Furthermore, for each $n \in \mathbb{N}$, we define a map $\pi_n : I_{n+1} \to I_n$ such that

$$\pi_n(\langle x_0, B_0\rangle, \cdots, \langle x_n, B_n\rangle, \langle x_{n+1}, B_{n+1}\rangle) = (\langle x_0, B_0\rangle, \cdots, \langle x_n, B_n\rangle)$$

for every finite σ-sequence $(\langle x_0, B_0\rangle, \cdots, \langle x_n, B_n\rangle, \langle x_{n+1}, B_{n+1}\rangle)$ in I_{n+1}. It can be checked readily that $(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ is an open sieve on X. We are to prove that this sieve satisfies (a). Let $(i_n)_{n<\omega}$ be any π-chain with $\bigcap_{n<\omega} U_{i_n} \neq \varnothing$. By the construction above, there exists a σ-sequence $(\langle x_n, B_n\rangle)_{n<\omega}$ in MP(X) corresponding to $(i_n)_{n<\omega}$ such that $\bigcap_{n<\omega} B_n \neq \varnothing$. Let $\mathfrak{F}$ be a filterbase controlled by $(U_{i_n})_{n<\omega}$. If $\mathfrak{F}$ does not clusters at anywhere, then for each point $p \in \bigcap_{n<\omega} B_n$ there exist an open neighborhood W_p of p and an $F_p \in \mathfrak{F}$ with $W_p \cap F_p = \varnothing$. Since $\bigcap_{n<\omega} B_n$ is compact, there are finitely many points $p_1, ..., p_k$ in $\bigcap_{n<\omega} B_n$ such that $W = \bigcup_{i=1}^k W_{p_i} \supseteq \bigcap_{n<\omega} B_n$. It follows that $W \cap (\bigcap_{i=1}^k F_{p_i}) = \varnothing$. Since σ is a winning strategy for α, then by condition (ii) in MP(X), we may select an $F \in \mathfrak{F}$ such that $F \subseteq (\bigcap_{i=1}^k F_{p_i}) \cap W$. However, this is a contradiction. $\square$

A regular space which satisfies any condition in Theorem 2.1 is called a *monotonically p-space*, abbrievated as *mp-space*. Moreover, a space is called *sieve complete* [4, 9] if there exists an open sieve such that $(U_{i_n})_{n<\omega}$ is a complete sequence for each π-chain $(i_n)_{n<\omega}$. Topsøe [14] characterized a sieve complete space X by using the game $\mathrm{SC}^{\mathrm{M}}(X)$. In $\mathrm{SC}^{\mathrm{M}}(X)$, β and α play the game as in CH(X), and α is said to *win* a play

$$\mathbf{P} = \{(x_n, B_n, A_n) : x_n \in A_n \subseteq B_n \text{ and } B_{n+1} \subseteq A_n \text{ for all } n < \omega\}$$

of $\mathrm{SC}^{\mathrm{M}}(X)$ if every net eventually in every B_n $(n < \omega)$ clusters in X.

Theorem 2.2. *The following are equivalent for a regular space X.*

(a) *X is sieve complete.*

(b) *Player α has a stationary winning strategy in $SC^M(X)$.*

(c) *Player α has a stationary winning strategy in MC(X).*

(d) *Player α has a winning strategy in $SC^M(X)$.*

(e) *Player α has a winning strategy in MC(X).*

Proof. The proofs of (a) $\Rightarrow$ (b) and (e) $\Rightarrow$ (a) are similar to the corresponding parts in Theorem 2.1, thus we omit them.

(b) $\Rightarrow$ (c). Suppose that α has a stationary winning strategy σ in $\mathrm{SC}^{\mathrm{M}}(X)$. We shall prove that σ is also a stationary winning for α in MC(X). Let $(\langle x_n, B_n\rangle)_{n<\omega}$ be any σ-sequence in $\mathrm{SC}^{\mathrm{M}}(X)$. By regularity of X, we may assume $\overline{B_{n+1}} \subseteq B_n$ for all $n < \omega$. Since $(x_n)_{n<\omega}$ is eventually in every B_n, it clusters. Thus, $\bigcap_{n<\omega} B_n \neq \varnothing$. Similarly, every net in $\bigcap B_n$ is eventually in every B_n, which means that $\bigcap_{n<\omega} B_n$ is compact. Let W be an open set containing $\bigcap_{n<\omega} B_n$. Suppose that for every $n < \omega$ there exists $z_n \in B_n \setminus W$. Then $(z_n)_{n<\omega}$ is a net eventually in each B_n, so it has a cluster point $z \notin W$. But, $z \in \bigcap_{n<\omega} B_n$, which leads to a contradiction. Hence, $B_n \subseteq W$ for some $n < \omega$. This implies that $(\langle x_n, B_n\rangle)_{n<\omega}$ is also a σ-sequence in $\mathrm{MC}(X)$.

(c) $\Rightarrow$ (d). Suppose that τ is a stationary winning strategy for α in $\mathrm{MC}(X)$. Let $(\langle x_n, B_n\rangle)_{n<\omega}$ be a τ-sequence in $\mathrm{MC}(X)$, and let $(y_\delta)_{\delta\in E}$ be a net which is eventually in each B_n. For each $\delta \in E$, we put $F_\delta = \{y_\lambda : \lambda \geq \delta\}$. Then the filterbase $(F_\delta)_{\delta\in E}$ is controlled by $(B_n)_{n<\omega}$. By an argument similar to that in Theorem 2.1, we can show that $(F_\delta)_{\delta\in E}$ clusters, which implies that $(y_\delta)_{\delta\in E}$ has a cluster point in X. Thus, $(\langle x_n, B_n\rangle)_{n<\omega}$ is also a τ-sequence in $\mathrm{SC}^{\mathrm{M}}(X)$.

(d) $\Rightarrow$ (e). It is similar to (b) $\Rightarrow$ (c), so its proof is omitted. $\square$

3. Extensions of Choquet's results

Strong α-favorability and Čech completeness are not equivalent in general, simply because there exist locally completely metrizable (of course, strongly α-favorable) Moore spaces which are not Čech complete [4, 6].

Theorem 3.1. *A regular space X is sieve complete if and only if it is both strongly α-favorable and mp.*

Proof. The necessity follows from Theorem 2.2 directly. Suppose that X is both strongly α-favorable and mp. Let σ_1 and σ_2 be stationary winning strategies for Player α in $\mathrm{CH}(X)$ and $\mathrm{MP}(X)$ respectively. We define a stationary strategy σ_3 for α inductively as follows: Suppose that we have defined σ_3 for all finite sequences $(\langle x_0, B_0\rangle, ..., \langle x_i, B_i\rangle)$ in $\bigcup_{x\in X}\{x\} \times o(x)$, where $0 \leq i \leq n$, such that $x_j \in \sigma_1(\langle x_j, B_j\rangle) \subseteq B_j \subseteq \sigma_1(\langle x_{j-1}, B_{j-1}\rangle)$ and $x_j \in \sigma_2(\langle x_j, B_j\rangle) \subseteq B_j \subseteq \sigma_2(\langle x_{j-1}, B_{j-1}\rangle)$ for all $1 \leq j \leq i$, whenever $x_j \in \sigma_3(\langle x_j, B_j\rangle) \subseteq B_j \subseteq \sigma_3(\langle x_{j-1}, B_{j-1}\rangle)$ for all $1 \leq j \leq i$. Let

$$(\langle x_0, B_0\rangle, ..., \langle x_n, B_n\rangle, \langle x_{n+1}, B_{n+1}\rangle)$$

be any finite sequence in $\bigcup_{x\in X}\{x\} \times o(x)$ such that $x_i \in \sigma_3(\langle x_i, B_i\rangle) \subseteq B_i \subseteq \sigma_3(\langle x_{i-1}, B_{i-1}\rangle)$ for all $1 \leq i \leq n$, and $x_{n+1} \in B_{n+1} \subseteq \sigma_3(\langle x_n, B_n\rangle)$. Then we define $\sigma_3(\langle x_{n+1}, B_{n+1}\rangle) = \sigma_1(\langle x_{n+1}, B_{n+1}\rangle) \cap \sigma_2(\langle x_{n+1}, B_{n+1}\rangle)$. This completes the definition of σ_3. It is easy to see that each σ_3-sequence is also a σ_1-sequence and σ_2-sequence, so any σ_3-sequence will satisfy both condtions (i) and (ii) in $\mathrm{MC}(X)$. Thus, σ_3 is a stationary winning strategy for α in $\mathrm{MC}(X)$. $\square$

Corollary 3.2. *A Moore space is sieve complete if and only if it is strongly α-favorable.*

Sieve complete spaces are also called *monotonically Čech complete* in the lierature, e.g., [4]. For paracompact spaces, sieve completeness is equivalent to Čech completeness. Thus, Theorem 3.1 extends the first result of Choquet (i.e., (†) in Section 1). To extend the second result of Choquet, we need some notation. A family of subsets of a space X is said to be an *almost cover* if the union of all its members is dense in X. We shall call $(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ an *almost sieve* if it has all the properties of a sieve except that $\{U_i : i \in I_n\}$ need only be an almost cover of X and $\{U_j : j \in \pi_n^{-1}(i)\}$ need only be an almost cover of U_i for all $i \in I_n$ and all $n \in \omega$. A space X is called *almost complete* (*almost mp*) if X has an almost open sieve which is complete (i.e., satisfying the conclusion in Theorem 2.1 (a)). Every almost complete Tychonoff space has a Čech complete dense G_δ-subspace.

Theorem 3.3. *A regular space X is almost complete if and only if it is weakly α-favorable and almost mp.*

Proof. The necessity is clear by definition. So, we only need to prove the sufficiency. Suppose that $(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ is an almost open sieve such that for each π-chain $(i_n)_{n<\omega}$ with $\bigcap_{n<\omega} U_{i_n} \neq \varnothing$, $(U_{i_n})_{n<\omega}$ is a complete sequence, and let σ be a winning strategy for Player α in BM(X). We shall define inductively an almost open complete sieve. First of all, let $J_0 = \{j_0\}$ be an arbitrary singleton and $V_{j_0} = X$. Since $\bigcup_{i\in I_1} U_i$ is dense in X, to each nonempty open set $B_1 \subseteq X$, we could assign a nonempty open set $U_{i_{B_1}}$ such that $B_1 \cap U_{i_{B_1}} \neq \varnothing$, where $i_{B_1} \in I_1$. Let

$$J_1 = \{B_1 \cap U_{i_{B_1}} : B_1 \in o(X), i_{B_1} \in I_1\}.$$

For each $j_1 = B_1 \cap U_{i_{B_1}} \in J_1$, we define $V_{j_1} = \sigma(B_1 \cap U_{i_{B_1}})$, and $\pi'_0(j_1) = j_0$. Now, fix any $j_1 = B_1 \cap U_{i_{B_1}} \in J_1$. To each nonempty open set $B_2 \subseteq V_{j_1}$, we assign an open set $U_{i_{B_2}}$ such that $B_2 \cap U_{i_{B_2}} \neq \varnothing$, where $i_{B_2} \in \pi_1^{-1}(i_{B_1})$. This is possible, since $\bigcup\{U_i : i \in \pi_1^{-1}(i_{B_1})\}$ is dense in $U_{i_{B_1}}$. Next, we define

$$J_2 = \{(B_1 \cap U_{i_{B_1}}, B_2 \cap U_{i_{B_2}}) : B_1, B_2 \in o(X),\ B_2 \subseteq \sigma(B_1 \cap U_{i_{B_1}})\}.$$

For each $j_2 = (B_1 \cap U_{i_{B_1}}, B_2 \cap U_{i_{B_2}}) \in J_2$, define $V_{j_2} = \sigma(B_1 \cap U_{i_{B_1}}, B_2 \cap U_{i_{B_2}})$, and define $\pi'_1 : J_2 \to J_1$ by $\pi'_1(j_2) = B_1 \cap U_{i_{B_1}}$. It is easy to see that $\{V_j : j \in J_1\}$ and $\{V_j : j \in J_2\}$ are almost open covers of X, and $\{V_j : j \in {\pi'}_1^{-1}(j_1)\}$ is an almost open cover of V_{j_1} for each $j_1 \in J_1$. Continuing this procedure infinitely many times inductively, we can define an open almost sieve $(\{V_j : j \in J_n\}, \pi'_n)_{n<\omega}$. To see that this almost sieve is complete, let $(j_n)_{n<\omega}$ be a π'-chain. Then there exist a sequence $(B_n)_{n\in\mathbb{N}}$ of nonempty open sets and a π-chain $(i_{B_n})_{n<\omega}$ such that $(B_n \cap U_{i_{B_n}})_{n\in\mathbb{N}}$ is a σ-sequence in BM(X) and $V_{j_n} = \sigma(B_1 \cap U_{i_{B_1}}, ..., B_n \cap U_{i_{B_n}})$ for each $n \in \mathbb{N}$. Since σ is a winning strategy for Player α in BM(X), $\bigcap_{n<\omega} U_{i_{B_n}} \neq \varnothing$. By Theorem 2.1, $(U_{i_{B_n}})_{n<\omega}$ is a complete sequence, which in turn implies $(V_{j_n})_{n<\omega}$ is a complete sequence, as $V_{j_n} \subseteq U_{i_{B_n}}$ for all $n \in \mathbb{N}$. □

Corollary 3.4. *An mp-space is almost complete if and only if it is weakly α-favorable.*

White [15] proved that a weakly α-favorable space with a base of countable order (abbreviated as BCO), is almost complete. Since every metric space has a base of countable order, and each space with a base of countable order is an mp-space, Theorem 3.3 extends both the second result of Choquet (i.e., (‡) in Section 1), and the previously mentioned result of White. Also, we notice that the Sorgenfrey line is strongly α-favorable, but it is not an almost mp-space, since it is not almost complete.

4. Nearly continuous and δ-open mappings

In Functional Analysis, Closed Graph Theorem and Open Mapping Theorem of linear mappings between Banach spaces are two fundamental and important theorems. Many extensions of these two classical theorems for mappings between topologically complete spaces have been established in the literature,, where the lack of linearity of a mapping can be compensated by the notion of near continuity or near openness, see [2, 3, 11] and [12]. In [12], the concept of separating maps is used to generalize the main theorem in [3]. Our first theorem in this section extends the main result in [12] to sieve complete spaces by a similar argument. Recall that a mapping $f : X \to Y$ is *nearly continuous* if for every $x \in X$ and every neighborhood V of $f(x)$, the set $\overline{f^{-1}(V)}$ is a neighborhood of x. Dually, f is said to be *nearly open* if for every $x \in X$ and every neighborhood U of x, the set $\overline{f(U)}$ is a neighborhood of $f(x)$. A mapping $f : X \to Y$ is *separating* if for any two distinct points $u, v \in Y$ there exist open neighborhoods U, V of u, v respectively such that $f^{-1}(U)$ and $f^{-1}(V)$ are separated sets.

Theorem 4.1. *If $f : X \to Y$ is a nearly continuous and separating mapping into a sieve complete regular space Y, then f is continuous.*

Proof. Let $(\{U_i : i \in I_n\}, \pi_n)_{n<\omega}$ be an open sieve on Y. Suppose that f is not continuous at some point $x \in X$. Then there is an open neighborhood V of $f(x)$ such that $f(U) \setminus V \neq \varnothing$ for every neighborhood U of x. Select an open neighborhood G of $f(x)$ with $\overline{G} \subseteq V$. Since f is nearly continuous, $\overline{f^{-1}(G)}$ is a neighborhood of x. It follows that there exists some point $y \in \overline{f^{-1}(G)} \cap f^{-1}(Y \setminus \overline{G})$. Next, select an open neighborhood H of $f(y)$ such that $\overline{H} \cap \overline{G} = \varnothing$. Again, by near continuity of f, there exists a point $x_0 \in \overline{f^{-1}(H)} \cap f^{-1}(G)$. Now, select an $i_0 \in I_0$, and an open neighborhood A_0 of $f(x_0)$ such that $\overline{A_0} \subseteq G \cap U_{i_0}$. Since $\overline{f^{-1}(A_0)} \cap f^{-1}(H) \neq \varnothing$, we can choose a point $y_0 \in \overline{f^{-1}(A_0)} \cap f^{-1}(H)$, an open neighborhood B_0 of y_0, and an $i'_0 \in I_0$ such that $\overline{B_0} \subseteq H \cap U_{i'_0}$. Repeat this procedure infinitely many times, we obtain two π-chains $(i_n)_{n<\omega}$ and $(i'_n)_{n<\omega}$, two sequences $(A_n)_{n<\omega}$ and $(B_n)_{n<\omega}$ of open sets in Y such that for every $n < \omega$, $\overline{A_{n+1}} \subseteq A_n \cap U_{i_n}$, $\overline{B_{n+1}} \subseteq B_n \cap U_{i'_n}$ and $\overline{f^{-1}(B_n)} \cap f^{-1}(A_n) \neq \varnothing$. Since $(U_{i_n})_{n<\omega}$ and $(U_{i'_n})_{n<\omega}$ are two complete sequences, then $\bigcap_{n<\omega} A_n$ and $\bigcap_{n<\omega} B_n$ are nonempty compact and disjoint sets. Moreover, $\{A_n : n < \omega\}$ and $\{B_n : n < \omega\}$ are outer bases for $\bigcap_{n<\omega} A_n$ and $\bigcap_{n<\omega} B_n$ respectively. As f is separating, there are open sets U and V such that $\bigcap_{n<\omega} A_n \subseteq U$ and $\bigcap_{n<\omega} B_n \subseteq V$, $f^{-1}(U)$ and $f^{-1}(V)$ are separated. Select some $n_0 < \omega$ such that $A_{n_0} \subseteq U$ and $B_{n_0} \subseteq V$. This implies that $f^{-1}(A_{n_0})$ and $f^{-1}(B_{n_0})$ are separated, which is a contradiction. □

Example 4.2. *There exists a non-continuous, nearly continuous and separating mapping $f : X \to Y$ of a metric space X into an almost complete Hausdorff space Y.* Let X be the set of all real numbers with the usual Euclidean topology. Let Y be the set of all real numbers endowed with the topology generated by the subbase $\{U \subseteq Y : U \text{ is open in } X\} \cup \{\mathbb{P}\}$, where $\mathbb{P}$ is the set of all irrational numbers, and $f : X \to Y$ is the identity mapping. Clearly, Y is Hausdorff. Let $\mathbb{Q}$ be the set of all rational numbers. Since any irrational number and the closed subset $\mathbb{Q}$ cannot be separated by open subsets, the space Y is not regular. Note that $\mathbb{P}$ is a Čech-complete dense G_δ-subspace of Y, thus Y is almost complete. It can be checked easily that f is nearly continuous and separating. But f is not continuous, as $f^{-1}(\mathbb{P})$ is not open in X. □

There are some versions of Open Mapping Theorem corresponding to Theorem 4.1. For instance, Noll [10] has proved that every nearly open continuous bijection from an almost complete regular space to a Hausdorff space is open. In studying images, preimages of Baire spaces and relevant properties, two types of mappings called feebly open mappings and δ-open mappings have played important roles. Recall that a mapping $f : X \to Y$ of a space X to a space Y is *feebly open* (*δ-open*) if $\mathrm{Int} f(U) \neq \varnothing$ ($\mathrm{Int}\overline{f(U)} \neq \varnothing$) for each nonempty open $U \subseteq X$ [8, 9]. It is clear that every feebly open mapping is δ-open. We are interested in the question when δ-open mappings are feebly open. We call a space X *quasi-regular* if for every nonempty open set U of X there is an nonempty open set V with $\overline{V} \subseteq U$.

Theorem 4.3. *Every δ-open continuous bijection $f : X \to Y$ of an almost complete quasi-regular space X onto a regular space Y is feebly open.*

Proof. Let $U \subseteq X$ be a nonempty open subset. By quasi-regularity of X, we can choose a nonempty open subset V such that $\overline{V} \subseteq U$. Since f is continuous and δ-open, $V \cap f^{-1}(\mathrm{Int}\overline{f(V)})$ is nonempty open. Hence, we may assume that $f(V) \subseteq \mathrm{Int}\overline{f(V)}$. We shall show $\mathrm{Int}\overline{f(V)} \subseteq f(U)$. To do so, let $y \in \mathrm{Int}\overline{f(V)}$ be an arbitrary point. Pick a point $x \in X$ with $y = f(x)$. It suffices to show $x \in \overline{V}$. Let W be any open neighborhood of x. We are left to prove $V \cap W \neq \varnothing$. Again, by continuity of f, we may assume $f(W) \subseteq \mathrm{Int}\overline{f(V)}$. Let $V_1 = V \cap f^{-1}(\mathrm{Int}\overline{f(W)})$, and $W_1 = W \cap f^{-1}(\mathrm{Int}\overline{f(W)})$. Then $f(V_1)$ and $f(W_1)$ are dense subsets of $\mathrm{Int}\overline{f(W)}$. Note that open subspaces of an almost complete space are also almost complete, and restrictions of a δ-open mapping on open subspaces of its domain are still δ-open. Thus, V_1 and W_1 are almost complete, $f \restriction V_1 : V_1 \to \mathrm{Int}\overline{f(W)}$ and $f \restriction W_1 : W_1 \to \mathrm{Int}\overline{f(W)}$ are continuous and δ-open. By Proposition 6.5 of [9], $f(V_1)$ and

$f(W_1)$ are almost complete. Since X is quasi-regular and almost complete, similar to Proposition 4.5 of [9], X is a Baire space. Thus, Y is also a Baire one, and so is $\mathrm{Int}\overline{f(W)}$. It is easily seen that both $f(V_1)$ and $f(W_1)$ are second category subsets of $\mathrm{Int}\overline{f(W)}$, which implies that $f(V_1) \cap f(W_1) \neq \varnothing$. Since f is injective, $V_1 \cap W_1 \neq \varnothing$. Hence, $V \cap W \neq \varnothing$. □

A mapping $f : X \to Y$ is called *base ic-continuous* if there exists a base $\mathcal{B}$ of Y such that $\mathrm{Int}f^{-1}(V) = f^{-1}(V) \cap \mathrm{Int}\overline{f^{-1}(V)}$ for each $V \in \mathcal{B}$. It is shown in [7] that every homomorphism between two topological groups is base *ic*-continuous. Similarly, one can show that linear mappings between Banach spaces are base *ic*-continuous. It is known that linear mappings between Banach spaces are nearly continuous. Motivated by these facts, one may ask whether base *ic*-continuity has any relations with near continuity in the realm of certain "topologically complete" spaces. Unfortunately, the answer is "no". Recall that a space is *resolvable* if it is dense-in-itself and has two disjoint dense subsets. It is well-known that dense-in-itself sequential spaces are resolvable.

Theorem 4.4. *Let X be any resolvable and connected Hausdorff space. Then there exists a nearly continuous real-valued mapping $f : X \to \mathbb{R}$ which is not base ic-continuous.*

Proof. Let D_1 and D_2 be two disjoint dense subsets of X. Without loss of generality, we may assume that $X = D_1 \cup D_2$. Since X is Hausdorff, we can choose a nonempty open subset G of X such that $\overline{G} \neq X$. Let $f : X \to \mathbb{R}$ be a real-valued mapping defined by the following formula

$$f(x) = \begin{cases} 1, & x \in \overline{G}; \\ 1, & x \in (X \setminus \overline{G}) \cap D_1; \\ 0, & \text{Otherwise.} \end{cases}$$

We are going to show that f is nearly continuous. First, we notice that

$$f^{-1}(\{0\}) = (X \setminus \overline{G}) \cap D_2 \subseteq X \setminus \overline{G} \subseteq \mathrm{Int}\overline{f^{-1}(\{0\})}.$$

Second, as $X = \overline{G} \cup (X \setminus \overline{G}) \subseteq \overline{f^{-1}(\{1\})}$, we obtain $f^{-1}(\{1\}) \subseteq \mathrm{Int}\overline{f^{-1}(\{1\})}$. Thus, f is nearly continuous. Furthermore, note that $\mathrm{Int}f^{-1}(\{1\}) = \mathrm{Int}\overline{G}$ and $\overline{G} = f^{-1}(\{1\}) \cap \overline{\mathrm{Int}f^{-1}(\{1\})}$. As X is connected, $\overline{G}$ is not open. Thus, we have $\overline{G} \neq \mathrm{Int}\overline{G}$. It follows that $\mathrm{Int}f^{-1}(\{1\}) \neq f^{-1}(\{1\}) \cap \overline{\mathrm{Int}f^{-1}(\{1\})}$. Therefore, f is not base *ic*-continuous. □

By Theorem 4.4, there are many nearly continuous functions which are not base *ic*-continuous. On the other hand, the simple mapping $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, is separating, base *ic*-continuous and has a closed graph. But, it is not nearly continuous.

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