THE CARDINALITY OF THE CLASS OF SEPARATELY CONTINUOUS FUNCTIONS

Zbigniew Piotrowski — Ivan L. Reilly — Brian M. Scott

ABSTRACT. Certain conditions on topological spaces $X$ and $Y$ are given so that the class of all separately continuous real-valued functions from $X \times Y$ has cardinality $c$, it is in particular when $X \times Y$ is the plane $\mathbb{R}^2$. On the other hand, it is shown that a closely related class of symmetrically quasi-continuous functions on $\mathbb{R}^2$ has cardinality $2^c$.

There are a few papers where an estimate is given for the cardinality of the class of continuous functions in terms of the weight or the density character of the domain space and/or the range of the function.

The following Proposition ([We] Lemma 3, p. 73) is an example of such result.

**Proposition 1.** Let $X$ be a topological space and let $Y$ be a $T_2$ space. Then $|C(X,Y)| \leq \pi(X)^{c(Y)\omega(Y)}$, where $c(X), \omega(X)$ and $\pi(X)$ denote the cellularity, the weight, and the $\pi$-weight of $X$, respectively, and $|C(X,Y)|$ denotes the cardinality of the class of continuous functions from $X$ to $Y$.

Observe that for $c(X) = \omega(Y) = \pi(X) = \omega$ we have

$$|C(X,Y)| \leq \omega^{\omega\cdot\omega} = \omega^\omega = 2^\omega.$$  

The following estimate [DDM, Corollary 8, 9 p. 68] is well-known.

**Proposition 2.** If $X$ is separable and satisfies the first axiom of countability, then $|C(X,\mathbb{R})| = c$.

We now give a rough estimate of the cardinality of separately continuous real-valued functions defined on the unit square.

A "naive proof" showing that $2^c$ is an upper bound could proceed as follows:

First, consider $x$-sections. There are $c$-many points in $[0, 1]$. Every point $x_0 \in [0, 1]$ generates a fibre $\{x_0\} \times Y$. Now $\{x_0\} \times Y$ is homeomorphic to $Y$. Since there are $c$-many continuous functions from $Y = [0, 1]$ into $\mathbb{R}$, we have $c$-many

---

**AMS Subject Classification (1991):** 54C30, 26B05.

**Key words:** separate continuity, quasi-continuity.
We now give a rough estimate of the cardinality of separately continuous real-valued functions defined on the unit square.

A "naive proof" showing that \(2^c\) is an upper bound could go as follows:

First, consider \(x\)-sections. There are \(c\)-many points in \([0, 1]\). Every point \(x_0 \in [0, 1]\) generates a fibre \(\{x_0\} \times Y\). Now \(\{x_0\} \times Y\) is homeomorphic to \(Y\). Since there are \(c\) many continuous functions from \(Y = [0, 1]\) into \(\mathbb{R}\) we have \(c\) many continuous functions on \(\{x_0\} \times Y\). So, altogether we have: \(c \cdot c \cdot \cdots \cdot c = c^c\). So there are at most \(2^c\) many continuous \(x\)-sections. Similarly, this "proof" shows also \(2^c\) many continuous \(y\)-sections. Clearly, what we see is that "not all" \(x\)-sections are "suitable" for the corresponding \(y\)-sections. So \(c^c\) is a rough estimate. We shall soon show that this bound can be lowered to \(c\), see [Si] p. 81. First we shall exhibit an explicit construction going back to R. Baire [Ba], see also W. Rudin [Ru]. We shall prove that all separately continuous functions \(f : \mathbb{R}^2 \to \mathbb{R}\) are of the first class of Baire. Let \(f\) be such a function. For each natural number \(n\), draw vertical lines in the plane, each at distance \(\frac{1}{n}\) from its left and right neighbors. Define \(f_n(x, y)\) to be \(f(x, y)\) on the union of these lines and determine \(f_n(x, y)\) on the rest of the plane by linear interpolation in the \(x\) variable. Since \(f\) is a continuous function of \(y\), for each fixed \(x\), each \(f_n\) is a continuous function on \(\mathbb{R}^2\). Since \(f\) is a continuous function of \(x\), for each fixed \(y\), \(\lim_{n \to \infty} f_n(x, y) = f(x, y)\), for all \((x, y) \in \mathbb{R}^2\). So we have obtained a separately continuous \(f\) as the limit of a (countable) sequence of continuous functions. Hence it follows that there are \(c\) many separately continuous functions, since there are \(c^c = c\) many sequences having terms from a set of cardinality \(c\).

**Remark 1.** The argument just presented can be easily generalized to any case where a separately continuous function is of the first class of Baire. However there are severe restrictions, e.g., metrizability of both spaces \(X\) and \(Y\), for this Baire - Lebesgue - Kuratowski - Montgomery Theorem, see [En] for example.

However the following result of Moran [Mo] is true.

**Proposition 3.** A separately continuous function \(f : X \times Y \to \mathbb{R}\) from a product \(X \times Y\) of compact spaces is the pointwise limit of a sequence of continuous functions on \(X \times Y\) if
and only if it is Baire measurable.

The reader is referred to [CK] and [Ve] for further generalizations.

**Proposition 4.** If $d(X), d(Y) \leq \kappa$ for some infinite cardinal $\kappa$, then there are at most $2^\kappa$ separately continuous functions $f : X \times Y \to \mathbb{R}$. (As usual, $d(X)$ denotes the density of $X$.)

**Proof.** Let $D$ and $E$ be dense subsets of $X$ and $Y$, respectively, each of cardinality at most $\kappa$. Suppose that $f, g : X \times Y \to \mathbb{R}$ are separately continuous and agree on $D \times E$.

Fix $x \in D; f$ and $g$ are continuous on $\{x\} \times Y$ and agree on its dense subset $\{x\} \times E$, so in fact they agree on $\{x\} \times Y$. Thus, $f$ and $g$ agree on $D \times Y$, and a second application of essentially the same argument shows that $f = g$.

Each separately continuous real-valued function on $X \times Y$ is therefore determined by its values on $D \times E$; and since there are at most $(2^\omega)^\kappa = 2^\kappa$ functions from $D \times E$ to $\mathbb{R}$, the result follows.

In fact the argument shows that if $d(X), d(Y) \leq \kappa$, and $Z$ is any Hausdorff space, then

$$\left| SC(X \times Y, Z) \right| \leq |Z|^\kappa,$$

where $SC(X \times Y, Z)$ is the set of separately continuous $f : X \times Y \to Z$. □

**Corollary 1.** There are $c$ many separately continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Nevertheless, there are spaces $X$ and $Y$ such that $|SC(X \times Y, \mathbb{R})| > |C(X \times Y, \mathbb{R})|$.

**Proposition 5.** Let $X$ be any $T_1$-space with $\kappa \geq \omega$ isolated points. Then $|SC(X \times X, \mathbb{R})| \geq 2^\kappa$.

**Proof.** Let $A$ be the set of isolated points of $X$. For each $\varphi : A \to \mathbb{R}$ define $f_\varphi : X \times X \to \mathbb{R}$ by

$$f_\varphi(x, y) = \begin{cases} \varphi(x), & \text{if } x = y \in A \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $f_\varphi$ is separately continuous. Since there are $c^\kappa = 2^\kappa$ such functions, the result is established. □
DEFINITION. Let \((X, <)\) be a fixed linearly ordered set. Then a set \(A\) is called cofinal in \(X\) if \(A \subseteq X\) and \(\forall x \in X \exists a \in A : x \leq a\). Further, the cofinality of \(X\), \(cf(X)\) is defined by:

\[
\text{cf}(X) = \min\{|A| : A \subseteq X \text{ is cofinal in } X\}
\]

EXAMPLES: \(\text{cf}(\mathbb{R}) = \omega, \text{cf}(\omega_\omega) = \omega\), since \(\bigcup_{n \in \omega} \omega_n = \omega\), and \(\text{cf}(\omega_1) = \omega_1\).

PROPOSITION 6. For any ordinals \(\alpha\) and \(\beta\), \(|C(\alpha \times \beta, \mathbb{R})| \leq \max\{2^\omega, |\alpha|^\omega, |\beta|^\omega\}\).

PROOF. We merely outline the proof, which is straightforward but tedious. The result is trivial for countable \(\alpha\) and \(\beta\). Suppose that \(\omega_1 \leq \alpha \leq \beta\) and that the result has been established for all products in which at least one factor is less than \(\alpha\). If \(\alpha = \tau + 1\), then by the induction hypothesis there are at most \(|C(\tau \times \beta, \mathbb{R})| \cdot |C(\{\tau\} \times \beta, \mathbb{R})| = |C(\tau \times \beta, \mathbb{R})| \cdot |C(1 \times \beta, \mathbb{R})| \leq |\beta|\) choices for \(f \in C(\alpha \times \beta, \mathbb{R})\). Similarly, if \(\text{cf}(\alpha) = \omega\), then \(\alpha\) is the discrete union of \(\omega\) subspaces each homeomorphic to an ordinal less than \(\alpha\), whence \(|C(\alpha \times \beta, \mathbb{R})| \leq (|\beta|^\omega)\omega = |\beta|^\omega\). We may therefore assume that \(\alpha\) is of uncountable cofinality, and the proof is completed by induction on \(\beta \geq \alpha\).

The arguments for \(\text{cf}(\beta) \leq \omega\) parallel those for \(\text{cf}(\alpha) \leq \omega\), so we assume further that \(\text{cf}(\beta)\) is uncountable. Fix \(f \in C(\alpha \times \beta, \mathbb{R})\).

CLAIM: There are \(\overline{\alpha} < \alpha\) and \(\overline{\beta} < \beta\) such that \(f\) is constant on \((\alpha \setminus \overline{\alpha}) \times (\beta \setminus \overline{\beta})\).

Once the CLAIM is established, the rest is easy, since \(\alpha \times \beta\) is the disjoint union of \((\alpha \setminus \overline{\alpha}) \times (\beta \setminus \overline{\beta}), \alpha \times \overline{\beta}, \) and \(\overline{\alpha} \times (\beta \setminus \overline{\beta})\). By hypothesis there are at most \(|\beta|^\omega\) continuous, real-valued functions on each of the last two sets; there are only \(|\alpha| \cdot |\beta|\) ways to choose \(\alpha\) and \(\beta\) and only \(2^\omega\) choices for the value of \(f\) on \((\alpha \setminus \overline{\alpha}) \times (\beta \setminus \overline{\beta})\), so altogether there are only at most \(|\beta|^\omega\) choices for \(f\).

The proof of the CLAIM relies heavily on the Pressing-Down Lemma ([Ju], Theorem A1.5). In case \(\text{cf}(\alpha) = \text{cf}(\beta)\) it is similar to the proof of the well-known result that any continuous, real-valued function on an ordinal of uncountable cofinality is eventually constant. If \(\text{cf}(\alpha) \neq \text{cf}(\beta)\), we must work a little harder. There is no harm in assuming that \(\text{cf}(\alpha) < \text{cf}(\beta)\). Let \(\langle \alpha_i : i < \text{cf}(\alpha)\rangle\) and \(\langle \beta_i : i < \text{cf}(\beta)\rangle\) be continuous, increasing sequences cofinal in \(\alpha\) and \(\beta\), respectively. For each \(\tau < \alpha, f|\{{\tau}\} \times \beta\) is eventually constant, so there are \(r_i \in \mathbb{R}(i < \text{cf}(\alpha))\) and \(\delta_0 < \beta\) such that \(f(\alpha_i, \delta) = r_i\) for each
$i < cf(\alpha)$ and $\delta \in \beta \setminus \delta_0$. Now fix $n \in \omega$ and a limit ordinal $i < cf(\alpha)$. If $\delta_0 \leq j < cf(\beta)$ and $j$ is a limit ordinal, then there are $a(j) < i$ and $b(j) < j$ such that if $\alpha_{a(j)} \leq \tau \leq a_i$ and $\beta_{b(j)} \leq \delta \leq \beta_j$, then $|f(\tau, \alpha) - \tau| < 2^{-n}$. It follows from the Pressing-Down Lemma that there are $\delta_i < \beta$ and a stationary $S \subseteq cf(\beta)$ such that $\beta_{b(j)} = \delta_i$ for each $j \in S$. And since $|S| \geq i$, there are $a(n, i) < i$ and a cofinal $Z_{n, i} \subseteq S$ such that $a(j) = a(n, i)$ for each $j \in Z_{n, i}$. Thus, if $\alpha_{a(n, i)} \leq \tau \leq \alpha_i$ and $\delta_i \leq \delta < \beta$, then $|f(\tau, \delta) - \tau| < 2^{-n}$. Let $\delta_n = \sup\{\delta_i : i < cf(\alpha)\} < \beta$, and note that by the Pressing-Down Lemma $a(n, i)$ is constant on a stationary set of $i < cf(\alpha)$. Thus $< \beta$, then $|f(\tau, \delta) - \tau| < 2^{1-n}$. To complete the argument it suffices to take $\bar{\alpha} = \sup\{a_{\bar{\alpha}(n)} : n \in \omega\}$ and $\bar{\beta} = \sup\{\delta_n : n \in \omega\}$.

Let $\gamma$ be an ordinal number and in $[0, \gamma]$ use the topology generated by all sets of form $\{x \in X \} \times \alpha \} \times \{x \in \beta \}$. We call this topological space the ordinal space $[0, \gamma]$, or simply, the ordinal space $\gamma$. Observe that the sets $(\alpha, \beta) = \{x \in X : \alpha \} \times \{x \in \beta \times \gamma \} \times \{x \in \beta + 1 \}$ are a basis for the topology.

**Example 1.** Let $X$ be the ordinal space $\mathbb{c}$. By Proposition 5 $|SC(X \times X, \mathbb{R})| \geq 2^\mathbb{c}$, and clearly equality must hold. By Proposition 6, however, $|C(X \times X, \mathbb{R})| \leq c^\mathbb{c} = \mathbb{c}$, and again it is clear that equality holds. Thus, $|C(X \times X, \mathbb{R})| = c < 2^\mathbb{c} = |SC(X \times X, \mathbb{R})|$; as one would expect, it is possible for a product to have strictly more (in the sense of cardinality) separately continuous real-valued functions than continuous ones.

According to Baire [Ba] it was Volterra who observed that if $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is separately continuous then for each point $(a, b) \in \mathbb{R} \times \mathbb{R}$, for each disc $D$ with center $(a, b)$ and for each $\varepsilon > 0$, there is a disc $D_1$ contained in $D$ such that $|f(x, y) - f(a, b)| < \varepsilon$ for all points $(x, y)$ in $D_1$. This property of separately continuous functions was subsequently called quasi-continuity by Kempisty [Ke] in 1932.

We say that a function $f : X \times Y \to Z$ is quasi-continuous with respect to $x$ if for each point $(x, y) \in X \times Y$, for each pair of open sets $U$ open in $X$ and $V$ open in $Y$ such that $(x, y) \in U \times V$, and for each open set $W$ in $Z$ such that $f(x, y) \in W$, there is an open set $U'$ in $X$ such that $x \in U' \subseteq U$ and a non-empty open set $V'$ in $Y$ such that $V' \subseteq V$ and
\[ f(U' \times V') \subset W. \]

Similarly, we define quasi-continuity with respect to \( y \). We say that \( f \) is symmetrically quasi-continuous if \( f \) is quasi-continuous with respect to \( x \) and quasi-continuous with respect to \( y \).

The following diagram is true, if \( X \) and \( Y \) are Baire, second countable spaces and \( f : X \times Y \to \mathbb{R} \), see [Ne] or [P1].

\[
\begin{array}{ccc}
\text{\( f \) is continuous} & \quad & \text{\( f \) is separately quasi-continuous} \\
\downarrow & \quad & \quad \downarrow \\
\text{\( f \) is symmetrically \( \rightarrow \)} & \quad & \text{\( f \) is separately \( \leftrightarrow \)} \\
\text{ quasi-continuous} & \quad & \text{ quasi-continuous} \\
\downarrow & \quad & \quad \downarrow \\
\text{\( f \) is quasi-continuous} & \quad & \text{\( f \) is quasi-continuous}
\end{array}
\]

In general, none of the above arrows can be reversed. As it can be seen, symmetric quasi-continuity is very close to separate continuity.

Now two questions arise.

(1) Can a symmetrically quasi-continuous function be determined from its values on a dense subset of its domain?

(2) What is the cardinality of the class of all symmetrically quasi-continuous functions defined on "nice" spaces, e.g., let \( X \times Y = \mathbb{R}^2 \)?

We shall answer both of these questions. Here is a negative answer to Question (1).

**Example 2.** Define \( f : \mathbb{R}^2 \to \mathbb{R} \) and \( g : \mathbb{R}^2 \to \mathbb{R} \) as follows:

\[
f(x, y) = \begin{cases} 
\sin \frac{1}{\sqrt{x^2 + y^2}} & \text{if } x^2 + y^2 \neq 0 \\
0, & \text{otherwise,}
\end{cases}
\]

and

\[
g(x, y) = \begin{cases} 
\sin \frac{1}{\sqrt{x^2 + y^2}} & \text{if } x^2 + y^2 \neq 0 \\
1, & \text{otherwise.}
\end{cases}
\]

We see that \( f \) and \( g \) agree on the entire plane, except for one point \((0, 0)\) and there they are different. \( \square \)

There are only \( c^c = (2^\omega)^c \) functions of any kind from \( \mathbb{R}^2 \) to \( \mathbb{R} \); we now show that there are \( 2^c \) symmetrically quasi-continuous functions on \( \mathbb{R}^2 \).
PROPOSITION 7. There are $2^c$ symmetrically quasi-continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

PROOF. Let us denote by $X_0$ the set $\{(x,y) : x > 0 \text{ and } \frac{1}{2}x < y < 2x\}$. Further, let $A = \{(x,y) : y = 2x \text{ and } x > 0\}$. Obviously, $\text{card } A = c$. Now, let us consider the power set $2^A$ of $A$, i.e., the set of all subsets of $A$. Clearly, $\text{card } 2^A = 2^c$. Let $A_t$ be the elements of $2^A$, $t \in T$. Now, let $\chi_t$ be the characteristic function of $X_0 \cup A_t$, i.e.,

$$
\chi_t(x, y) = \begin{cases} 
1, & \text{if } (x, y) \in X_0 \cup A_t \\
0, & \text{otherwise}
\end{cases}
$$

It is easy to see that there are $2^c$ functions $\chi_t$ and each one of them is symmetrically quasi-continuous. □

COROLLARY 2. There are $2^c$ many quasi-continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

ACKNOWLEDGEMENT: When this article was written the first-named author was 1993-94 YSU Research Professor.

REFERENCES


Z. Piotrowski,
Department of Mathematics

7
Youngstown State University,  
Youngstown, OH 44555, U.S.A.

I. L. Reilly,  
Department of Mathematics and Statistics,  
University of Auckland,  
Auckland, Private Bag, NEW ZEALAND

Brian M. Scott,  
Department of Mathematics,  
Cleveland State University,  
Cleveland, OH 44115, U.S.A.