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THE CARDINALITY OF THE CLASS OF SEPARATELY CONTINUOUS FUNCTIONS

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ABSTRACT. Certain conditions on topological spaces X and Y are given so that the class of all separately continuous real-valued functions from $X \times Y$ has cardinality $\mathfrak{c}$, it is in particular when $X \times Y$ is the plane $\mathbb{R}^2$. On the other hand, it is shown that a closely related class of symmetrically quasi-continuous functions on $\mathbb{R}^2$ has cardinality $2^{\mathfrak{c}}$.

There are a few papers where an estimate is given for the cardinality of the class of continuous functions in terms of the weight or the density character of the domain space and/or the range of the function.

The following Proposition ([We] Lemma 3, p. 73) is an example of such result.

PROPOSITION 1. *Let X be a topological space and let Y be a T_2 space. Then $|C(X,Y)| \le \pi(X)^{c(X)w(Y)}$, where $c(X), w(X)$ and $\pi(X)$ denote the cellularity, the weight, and the π-weight of X, respectively, and $|C(X,Y)|$ denotes the cardinality of the class of continuous functions from X to Y.*

Observe that for $c(X) = w(Y) = \pi(X) = \omega$ we have

$$|C(X,Y)| \le \omega^{\omega \cdot \omega} = \omega^{\omega} = 2^{\omega}.$$

The following estimate [DDM, Corollary 8, 9 p. 68] is well-known.

PROPOSITION 2. *If X is separable and satisfies the first axiom of countability, then $|C(X,\mathbb{R})| = \mathfrak{c}$.*

We now give a rough estimate of the cardinaity of separately continuous real-valued functions defined on the unit square.

A "naive proof" showing that $2^{\mathfrak{c}}$ is an upper bound could proceed as follows:

First, consider x-sections. There are $\mathfrak{c}$-many points in $[0,1]$. Every point $x_0 \in [0,1]$ generates a fibre $\{x_0\} \times Y$. Now $\{x_0\} \times Y$ is homeomorphic to Y. Since there are $\mathfrak{c}$-many continuous functions from $Y = [0,1]$ into $\mathbb{R}$, we have $\mathfrak{c}$-many

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A "naive proof" showing that 2^c is an upper bound could go as follows:

First, consider x-sections. There are c-many points in $[0, 1]$. Every point $x_0 \in [0, 1]$ generates a fibre $\{x_0\} \times Y$. Now $\{x_0\} \times Y$ is homeomorphic to Y. Since there are c many continuous functions from $Y = [0, 1]$ into $\mathbb{R}$ we have c many continuous functions on $\{x_0\} \times Y$. So, altogether we have: $\underbrace{c \cdot c \cdot \ldots c}_{c \text{ many times}} = c^c$. So there are at most 2^c many continuous x-sections. Similarly, this "proof" shows also 2^c many continuous y-sections. Clearly, what we see is that "not all" x-sections are "suitable" for the corresponding y-sections. So c^c is a rough estimate. We shall soon show that this bound can be lowered to c, see [Si] p. 81. First we shall exhibit an explicit construction going back to R. Baire [Ba], see also W. Rudin [Ru]. We shall prove that all separately continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ are of the first class of Baire. Let f be such a function. For each natural number n, draw vertical lines in the plane, each at distance $\frac{1}{n}$ from its left and right neighbors. Define $f_n(x, y)$ to be $f(x, y)$ on the union of these lines and determine $f_n(x, y)$ on the rest of the plane by linear interpolation in the x variable. Since f is a continuous function of y, for each fixed x, each f_n is a continuous function on $\mathbb{R}^2$. Since f is a continuous function of x, for each fixed y, $\lim_{n \to \infty} f_n(x, y) = f(x, y)$, for all $(x, y) \in \mathbb{R}^2$. So we have obtained a separately continuous f as the limit of a (countable) sequence of continuous functions. Hence it follows that there are c many separately continuous functions, since there are $c^\omega = c$ many sequences having terms from a set of cardinality c.

REMARK 1. The argument just presented can be easily generalized to any case where a separately continuous function is of the first class of Baire. However there are severe restrictions, e.g., metrizability of *both* spaces X and Y, for this Baire – Lebesgue – Kuratowski – Montgomery Theorem, see [En] for example.

However the following result of Moran [Mo] is true.

PROPOSITION 3. A separately continuous function $f : X \times Y \to \mathbb{R}$ from a product $X \times Y$ of compact spaces is the pointwise limit of a sequence of continuous functions on $X \times Y$ if and only if it is Baire measurable.

The reader is referred to [CK] and [Ve] for further generalizations.

PROPOSITION 4. If $d(X), d(Y) \leq \kappa$ for some infinite cardinal κ, then there are at most 2^κ separately continuous functions $f : X \times Y \to \mathbb{R}$. (As usual, $d(X)$ denotes the density of X.)

PROOF. Let D and E be dense subsets of X and Y, respectively, each of cardinality at most κ. Suppose that $f, g : X \times Y \to \mathbb{R}$ are separately continuous and agree on $D \times E$. Fix $x \in D$; f and g are continuous on $\{x\} \times Y$ and agree on its dense subset $\{x\} \times E$, so in fact they agree on $\{x\} \times Y$. Thus, f and g agree on $D \times Y$, and a second application of essentially the same argument shows that $f = g$.

Each separately continuous real-valued function on $X \times Y$ is therefore determined by its values on $D \times E$; and since there are at most $(2^\omega)^\kappa = 2^\kappa$ functions from $D \times E$ to $\mathbb{R}$, the result follows.

In fact the argument shows that if $d(X), d(Y) \leq \kappa$, and Z is any Hausdorff space, then

$$| SC(X \times Y, Z) | \leq | Z |^\kappa,$$

where $SC(X \times Y, Z)$ is the set of separately continuous $f : X \times Y \to Z$. □

COROLLARY 1. There are c many separately continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Nevertheless, there are spaces X and Y such that $| SC(X \times Y, \mathbb{R}) | > | C(X \times Y, \mathbb{R}) |$.

PROPOSITION 5. Let X be any T_1-space with $\kappa \geq \omega$ isolated points. Then $| SC(X \times X, \mathbb{R}) | \geq 2^\kappa$.

PROOF. Let A be the set of isolated points of X. For each $\varphi : A \to \mathbb{R}$ define $f_\varphi : X \times X \to \mathbb{R}$ by

$$f_\varphi(x, y) = \begin{cases} \varphi(x), & \text{if } x = y \in A \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that f_φ is separately continuous. Since there are $c^\kappa = 2^\kappa$ such functions, the result is established. □

DEFINITION. Let $(X,<)$ be a fixed linearly ordered set. Then a set A is called cofinal in X if $A \subset X$ and $\forall x \in X\ \exists a \in A : x \leq a$. Further, the cofinality of X, $cf(X)$ is defined by:

$$cf(X) = \min\{|A| \,\big|\, A \subset X \text{ is cofinal in } X\}$$

EXAMPLES: $cf(I\!R) = \omega, cf(\omega_\omega) = \omega$, since $\lim_{n \in \omega} \omega_n = \bigcup_{n \in \omega} \omega_n = \omega_\omega$, and $cf(\omega_1) = \omega_1$

PROPOSITION 6. For any ordinals α and β, $| C(\alpha \times \beta, \mathbb{R}) | \leq \max\{2^\omega, |\alpha|^\omega, | \beta |^\omega\}$.

PROOF. We merely outline the proof, which is straightforward but tedious. The result is trivial for countable α and β. Suppose that $\omega_1 \leq \alpha \leq \beta$ and that the result has been established for all products in which at least one factor is less than α. If $\alpha = \tau + 1$, then by the induction hypothesis there are at most $| C(\tau \times \beta, \mathbb{R}) | \cdot | C(\{\tau\} \times \beta, \mathbb{R})| =| C(\tau \times \beta, \mathbb{R}) | \cdot | C(1 \times \beta, \mathbb{R}) | \leq | \beta |$ choices for $f \in C(\alpha \times \beta, \mathbb{R})$. Similarly, if $cf(\alpha) = \omega$, then α is the discrete union of ω subspaces each homeomorphic to an ordinal less than α, whence $| C(\alpha \times \beta, \mathbb{R} | \leq (| \beta |^\omega)^\omega =| \beta |^\omega$. We may therefore assume that α is of uncountable cofinality, and the proof is completed by induction on $\beta \geq \alpha$.

The arguments for $cf(\beta) \leq \omega$ parallel those for $cf(\alpha) \leq \omega$, so we assume further that $cf(\beta)$ is uncountable. Fix $f \in C(\alpha \times \beta, \mathbb{R})$.

CLAIM: There are $\overline{\alpha} < \alpha$ and $\overline{\beta} < \beta$ such that f is constant on $(\alpha \backslash \overline{\alpha}) \times (\beta \backslash \overline{\beta})$.

Once the CLAIM is established, the rest is easy, since $\alpha \times \beta$ is the disjoint union of $(\alpha \backslash \overline{\alpha}) \times (\beta \backslash \overline{\beta}), \alpha \times \overline{\beta}$, and $\overline{\alpha} \times (\beta \backslash \overline{\beta})$. By hypothesis there are at most $| \beta |^\omega$ continuous, real-valued functions on each of the last two sets; there are only $| \alpha | \cdot | \beta | =| \beta |$ ways to choose α and β and only 2^ω choices for the value of f on $(\alpha \backslash \overline{\alpha}) \times (\beta \backslash \overline{\beta})$, so altogether there are only at most $| \beta |^\omega$ choices for f.

The proof of the CLAIM relies heavily on the Pressing-Down Lemma ([Ju], Theorem A1.5). In case $cf(\alpha) = cf(\beta)$ it is similar to the proof of the well-known result that any continuous, real-valued function on an ordinal of uncountable cofinality is eventually constant. If $cf(\alpha) \neq cf(\beta)$, we must work a little harder. There is no harm in assuming that $cf(\alpha) < cf(\beta)$. Let $\langle \alpha_i : i < cf(\alpha) \rangle$ and $\langle \beta_i : i < cf(\beta) \rangle$ be continuous, increasing sequences cofinal in α and β, respectively. For each $\tau < \alpha, f |(\{\tau\} \times \beta)$ is eventually constant, so there are $r_i \in \mathbb{R} (i < cf(\alpha))$ and $\delta_0 < \beta$ such that $f(\alpha_i, \delta) = r_i$ for each

$i < cf(\alpha)$ and $\delta \in \beta \backslash \delta_0$. Now fix $n \in \omega$ and a limit ordinal $i < cf(\alpha)$. If $\delta_0 \leq j < cf(\beta)$ and j is a limit ordinal, then there are $a(j) < i$ and $b(j) < j$ such that if $\alpha_{a(j)} \leq \tau \leq a_i$ and $\beta_{b(j)} \leq \delta \leq \beta_j$, then $| f(\tau, \alpha) - r_i | < 2^{-n}$. It follows from the Pressing-Down Lemma that there are $\delta_i < \beta$ and a stationary $S \subseteq cf(\beta)$ such that $\beta_{b(j)} = \delta_i$ for each $j \in S$. And since $| S | > | i |$, there are $a(n,i) < i$ and a cofinal $Z_{n,i} \subseteq S$ such that $a(j) = a(n,i)$ for each $j \in Z_{n,i}$. Thus, if $\alpha_{a(n,i)} \leq \tau \leq \alpha_i$ and $\delta_i \leq \delta < \beta$, then $| f(\tau, \delta) - r_i | < 2^{-n}$. Let $\delta_n = \sup\{\delta_i : i < cf(\alpha)\} < \beta$, and note that by the Pressing-Down Lemma $a(n,i)$ is constant on a stationary set of $i < cf(\alpha)$. Thus $< \beta$, then $| f(\tau, \delta) - r_{i(n)} | < 2^{1-n}$. To complete the argument it suffices to take $\overline{\alpha} = \sup\{a_{i(n)} : n \in \omega\}$ and $\overline{\beta} = \sup\{\delta_n : n \in \omega\}$.

□

Let γ be an ordinal number and in $[0, \gamma]$ use the topology generated by all sets of form $\{x | x > \alpha\}$ and $\{x | x < \beta\}$. We call this topological space the *ordinal space* $[0, \gamma]$, or simply, the ordinal space γ. Observe that the sets $(\alpha, \beta] = \{x | x > \alpha\} \cap \{x | x < \beta + 1\}$ are a basis for the topology

EXAMPLE 1. Let X be the ordinal space c. By Proposition 5 $| SC(X \times X), \mathbb{R}) | \geq 2^c$, and clearly equality must hold. By Proposition 6, however, $| C(X \times X, \mathbb{R}) | \leq c^\omega = c$, and again it is clear that equality holds. Thus, $| C(X \times X, \mathbb{R}) | = c < 2^c = | SC(X \times X, \mathbb{R}) |$; as one would expect, it is possible for a product to have strictly more (in the sense of cardinality) separately continuous real-valued functions than continuous ones. □

According to Baire [Ba] it was Volterra who observed that if $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is separately continuous then for each point $(a, b) \in \mathbb{R} \times \mathbb{R}$, for each disc D with center (a, b) and for each $\epsilon > 0$, there is a disc D_1 contained in D such that $| f(x, y) - f(a, b) | < \epsilon$ for all points (x, y) in D_1. This property of separately continuous functions was subsequently called *quasi-continuity* by Kempisty [Ke] in 1932.

We say that a function $f : X \times Y \rightarrow Z$ is *quasi-continuous with respect to* x if for each point $(x, y) \in X \times Y$, for each pair of open sets U open in X and V open in Y such that $(x, y) \in U \times V$, and for each open set W in Z such that $f(x, y) \in W$, there is an open set U' in X such that $x \in U' \subset U$ and a non-empty open set V' in Y such that $V' \subset V$ and

$f(U' \times V') \subset W$.

Similarly, we define *quasi-continuity with respect to y*. We say that f is *symmetrically quasi-continuous* if f is quasi-continuous with respect to x and quasi-continuous with respect to y.

The following diagram is true, if X and Y are Baire, second countable spaces and $f : X \times Y \to \mathbb{R}$, see [Ne] or [P1].

$$\begin{tikzcd}
 & f \text{ is continuous} \arrow[dl] \arrow[d] \arrow[dr] & \\
\begin{array}{c} f \text{ is symmetrically} \\ \text{quasi-continuous} \end{array} \arrow[dr] & \begin{array}{c} f \text{ is separately} \\ \text{continuous} \end{array} \arrow[l] \arrow[r] \arrow[d] & \begin{array}{c} f \text{ is separately} \\ \text{quasi-continuous} \end{array} \arrow[dl] \\
 & f \text{ is quasi-continuous} &
\end{tikzcd}$$

In general, none of the above arrows can be reversed. As it can be seen, symmetric quasi-continuity is very close to separate continuity.

Now two questions arise.

(1) Can a symmetrically quasi-continuous function be determined from its values on a dense subset of its domain?

(2) What is the cardinality of the class of all symmetrically quasi-continuous functions defined on "nice" spaces, e.g., let $X \times Y = \mathbb{R}^2$?

We shall answer *both* of these questions. Here is a negative answer to Question (1).

EXAMPLE 2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x,y) = \begin{cases} \sin \frac{1}{\sqrt{x^2+y^2}} & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x,y) = \begin{cases} \sin \frac{1}{\sqrt{x^2+y^2}} & \text{if } x^2 + y^2 \neq 0 \\ 1, & \text{otherwise.} \end{cases}$$

We see that f and g agree on the entire plane, except for one point $(0,0)$ and there they are different. □

There are only $c^c = (2^\omega)^c$ functions of any kind from $\mathbb{R}^2$ to $\mathbb{R}$; we now show that there are 2^c symmetrically quasi-continuous functions on $\mathbb{R}^2$.

PROPOSITION 7. There are 2^c symmetrically quasi-continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

PROOF. Let us denote by X_0 the set $\{(x,y) : x > 0 \text{ and } \frac{1}{2}x < y < 2x\}$. Further, let $A = \{(x,y) : y = 2x \text{ and } x > 0\}$. Obviously, card $A = c$. Now, let us consider the power set 2^A of A, i.e., the set of all subsets of A. Clearly, card $2^A = 2^c$. Let A_t be the elements of $2^A, t \in T$. Now, let χ_t be the characteristic function of $X_0 \cup A_t$, i.e.,

$$\chi_t(x,y) = \begin{cases} 1, & \text{if } (x,y) \in X_0 \cup A_t \\ 0, & \text{otherwise} \end{cases}.$$

It is easy to see that there are 2^c functions χ_t and each one of them is symmetrically quasi-continuous. □

COROLLARY 2. There are 2^c many quasi-continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$.

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