Closed Graph Theorem: Topological Approach

by

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Abstract. Some new types of Closed Graph Theorem are presented. These results generalize some theorems of T. Byczkowski, R. Pol and M. Wilhelm. An answer to a problem of M. Wilhelm is provided.

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Introduction. We say that a function f from a space X into a space Y has a closed graph if the graph of the function f, i.e. the set $\{(x,y) \in X \times Y : y = f(x)\}$, is a closed subset of the product $X \times Y$. It is well known that any continuous function f from a space X into a Hausdorff space Y has a closed graph. S. Banach [2] showed that any closed graph linear operator between any two Banach spaces is continuous. It transpires that the lack of linearity of f may be compensated by a near continuity in topological case.

Consider the following statement concerning the continuity of a function f

(*) If f is a nearly continuous closed graph function from a Hausdorff space X into a Čech-complete space Y then f is continuous.

It turns out that (*) holds if either one of the following conditions is true:

(a) X is metric, Čech-complete and Y is metric, see [6]

(b) f is compact, i.e. inverse images of compact sets are compact, see [3]

(c) f has a Δ-closed graph, [7]

In this paper we shall establish new types of Closed Graph Theorem. For this we introduce and consider what we call the separating function.

Our results generalize the previously mentioned theorems of T. Byczkowski and R. Pol [3] and M. Wilhelm [7], as well as allowing us to answer in the negative, the problem of M. Wilhelm [8]. Finally a Blumberg type theorem for separating functions is obtained.

1. Basic facts: We shall start by formulating and proving some facts concerning closed graph functions. We do this for two reasons: first, we will use them frequently in the sequel, and secondly we have not been able to find these results in just the form we needed. The interested reader may consult the survey paper [4].

Theorem 1. Let f be a function from a space X into a space Y. Then the following conditions are equivalent.

(a) f has a closed graph;

(b) if $x \in X$, $y \in Y$ and $y \neq f(x)$, then there exists an open neighborhood U of x and there exists an open neighborhood V of y such that $f(U) \cap V = \emptyset$;

(c) if K is a compact subset of X, then $f(K) = \bigcap\{\mathrm{cl}\ f(U)\colon U$ is an open neighborhood of $K\}$;

(d) if C is a compact subset of Y, then $f^{-1}(C) = \bigcap\{\mathrm{cl}\ f^{-1}(V)\colon V$ is an open neighborhood of $C\}$.

Proof: (a) $\leftrightarrow$ (b) is trivial, and follows immediately from the definition of the product topology. Note that (b) can be rewritten in the following forms: $\{f(x)\} = \bigcap\{\mathrm{cl}\ f(U)\colon U$ is an open neighborhood of $x\}$ for each $x \in X$; or $f^{-1}(y) = \bigcap\{\mathrm{cl}\ f^{-1}(V)\colon V$ is an open neighborhood of $y\}$ for each $y \in Y$. Since compact sets "behave" like points, we obtain (b) $\leftrightarrow$ (c) and (b) $\leftrightarrow$ (d). $\square$

Corollary 1. Let $f\colon X \to Y$ have a closed graph. Then

(a) if K is a compact subset of X, then f(K) is closed;

(b) if C is a compact subset of Y, then $f^{-1}(C)$ is closed;

(c) if K is a compact subset of X, C is a compact subset of Y and $f(K) \cap C = \emptyset$, then there is an open neighborhood U of K and an open neighborhood V of C such that $\mathrm{cl}\ f^{-1}(V) \cap U = \emptyset = V \cap \mathrm{cl}\ f(U)$.

Proof: For (a) use (c) from Theorem 1; for (b) use (d) from Theorem 1; for (c) use both (c) and (d) from Theorem 1. □

Corollary 2. Let $f: X \to Y$ have a closed graph.

(a) If X is compact, then f is a closed function

(b) If Y is compact, then f is a continuous function.

Proof: Because closed subsets of a compact space are compact, the result follows immediately from Corollary 1. □

A function $f: X \to Y$ is said to be *continuous at a point* $x \in X$ if for every open set $V \subset Y$ containing $f(x)$ there exists in X a neighborhood U of x such that $f(U) \subset V$.

A function $f: X \to Y$ is said to be *closed at a point* $y \in Y$ if for every open set $U \subset X$ which contains $f^{-1}(y)$ there exists in Y a neighborhood V of y such that $f^{-1}(V) \subset U$. This is the same as saying that $y \to f^{-1}(y)$ is an upper semicontinuous map.

Corollary 3. Let $f: X \to Y$ have a closed graph.

(a) If X is locally compact, then the set $CL_c(f) = \{y \in Y: f$ is closed at y and $f^{-1}(y)$ is compact$\}$ is an open subset of Y.

(b) If Y is locally compact, then the set $C(f) = \{x \in X: f$ is continuous at $x\}$ is an open subset of X. Moreover, if Y is regular, then the set $D(f)$ of points of discontinuity of f can be covered by $w(Y)$ or less nowhere dense closed subsets of X.

Proof: (a) Let $y \in CL_c(f)$. Since X is locally compact and $f^{-1}(y)$ is compact, there exists an open neighborhood U of $f^{-1}(y)$ such that cl U is compact. Since f is closed at y, there exists in Y an open neighborhood V of y such that $f^{-1}(V) \subset U$. We shall show that $V \subset CL_c(f)$. Let $v \in V$. Then, by virtue of Corollary 1 (b), $f^{-1}(v)$ is closed in X. Since $f^{-1}(v)$ is a subset of the compact space cl U, it has to be compact. We have to show also that f is closed at v. So let W be an open neighborhood of $f^{-1}(v)$. Since $f^{-1}(v)$ is contained in U we may assume that W is contained in U, as well. The restriction of f to cl U, $f|\mathrm{cl}\ U$, is a function from cl U to Y and has a closed graph. Since the domain of this function is a compact space, it is a closed function by Corollary 2 (a).

Thus it is closed at any point of the space Y. Hence there exists an open neighborhood G of v such that $(f|\mathrm{cl}\ U)^{-1}(G) \subset W$. We may assume that $G \subset V$ (by taking $G \cap V$ if needed). But then $(f|\mathrm{cl}\ U)^{-1}(G) = f^{-1}(G)$ and we are done.

(b) Let $x \in C(f)$. Let V be an open neighborhood of f(x) such that cl V is compact. Then there exists an open neighborhood U of X such that $U \subset f^{-1}(V)$. We shall show that $U \subset C(f)$. $f|U$ goes from U to the compact space cl V and has a closed graph. Therefore, it is a continuous function by virtue of Corollary 2 (b). Since $U \subset f^{-1}(V)$ and U, V are open, the continuity of the function $f|U$ means the same as the continuity of f. Hence f is continuous at each point of U, so $U \subset C(f)$.

To prove the second part of (b), take a base $\mathcal{B}$ in Y such that $|\mathcal{B}| = w(Y)$ and cl V is compact for each $V \in \mathcal{B}$. Because f has a closed graph, for each $V \in \mathcal{B}$ the set $f^{-1}(\mathrm{cl}\ V) - \mathrm{int}\ f^{-1}(\mathrm{cl}\ V)$ is closed and nowhere dense in X, (use (b) of Corollary 1). But trivially, each point of D(f) lies in some such set and we are done. □

<u>Example</u>. Let i denote the identity function from the unit interval I onto itself and let $\tilde{I}$ denote the same interval with the discrete topology The functions $i : \tilde{I} \to I$ and $i : I \to \tilde{I}$ have the closed graphs (use, for example, the arguments from Lemma 1 (b), (c)). The first of these acts on a locally compact space $\tilde{I}$ (to the metric compact space I), but $CL_c(i) = \emptyset$;

the second (acts on a compact metric space and) takes the values in a locally compact space $\tilde{I}$ but $C(i) = \emptyset$. These examples show that one cannot guess $CL_c(f) \neq \emptyset \neq C(f)$ in Corollary 3. However from the second part of (b) in Corollary 3, one can deduce that for any closed graph function from a space satisfying the Baire Category Theorem into a regular locally compact space with countable base, the set of points of continuity of such a function always contains a dense open subset. Moreover, any closed nowhere dense subset of reals can be the set of points of discontinuity of a real closed graph function [see for example [1]].

The next theorem indicates where the failure of discontinuity of a closed graph function lies.

A space X is said to be a *k-space* if for each $A \subset X$, the set A is closed in X provided that the intersection of A with any compact subspace Z of the space X is closed in Z.

The class of k-spaces is quite wide. It contains for example all Čech-complete spaces as well as all first countable spaces.

Theorem 4. Let $f: X \to Y$ have closed graph.

(a) If Y is a k-space and $f^{-1}(C)$ is compact whenever C is a compact subspace of Y, then f is a closed function.

(b) If X is a k-space, then f is a continuous function if and only if f(K) is compact whenever K is a compact subspace of X.

Proof (a): Let E be a closed subset of X and let C be a compact subspace of Y. Then the set $E \cap f^{-1}(C)$ is a closed subset of the compact space $f^{-1}(C)$ and therefore is compact. By Corollary 1 (a), $f(E \cap f^{-1}(C))$ is a closed subset of Y. But $f(E \cap f^{-1}(C)) = f(E) \cap C$, so $f(E) \cap C$ is also a closed subset of C. Since Y is a k-space, f(E) is a closed subset of the space Y.

(b) Only the "if" part needs proof. So let F be a closed subset of the space Y and let K be a compact subspace of the space X. Then the set $F \cap f(K)$ is a closed subset of the compact subspace $f(K)$ and therefore is compact. By Corollary 1 (b), $f^{-1}(F \cap f(K))$ is a closed subset of X and therefore $K \cap f^{-1}(F \cap f(K))$ is a closed subset of the space K. But $K \cap f^{-1}(F \cap f(K)) = K \cap f^{-1}(F) \cap f^{-1}(f(K)) = K \cap f^{-1}(F)$. Hence $K \cap f^{-1}(F)$ is closed in K. Since X is a k-space, $f^{-1}(F)$ is a closed subset of the space X. □

We have seen that compactness guarantees continuity of closed graph functions. Some other topological properties can do the same.

<u>Theorem 5</u>. If $f: X \to Y$ has the closed graph where X is a Fréchet space and Y is a countably compact space, then f is continuous.

<u>Proof</u>. Assume to the contrary that there exists a point $p \in X$ and an open neighborhood V of $f(p)$ such that $(f(U) - V \neq \emptyset$ whenever U is an open neighborhood of p. Let $F = Y - V$. Then $p \in \mathrm{cl}\, f^{-1}(F) - f^{-1}(F)$. Since X is a Fréchet space, there exists a sequence $\{x_n\}$ of points of the set $f^{-1}(F)$ converging to p. Because F is a closed subset of a countably compact space Y and the sequence $\{f(x_n)\}$ is contained in F, there exists a point $g \in F$ such that $g \in E$ whenever E is a closed subset of Y containing all but finitely many members of the sequence $\{f(x_n)\}$. Hence $g \in \mathrm{cl}\, f(U)$ for each open neighborhood U of p. In consequence, $g \in \bigcap \{\mathrm{cl}\, f(U) : U$ is an open neighborhood of $p\}$. However this is impossible in view of Theorem 1 (c). □

<u>2. Separating functions</u>. In this section we will attempt to find additional properties of spaces and functions that will guarantee the continuity of closed graph functions. For this purpose we distinguish a class of functions, which is a strenghtened version of closed graph functions, namely

the class of separating functions which we define as follows.

We say that a function f from a space X into a space Y is *separating* if it satisfies the following condition:

(*) any two distinct points $u, v \in Y$ there correspond respective open neighborhoods U and V such that the sets $f^{-1}(U)$, $f^{-1}(V)$ are separated, i.e., $f^{-1}(U) \cap \mathrm{cl}\, f^{-1}(V) = \emptyset = \mathrm{cl}\, f^{-1}(U) \cap f^{-1}(V)$.

The fact that each separating function has a closed graph may be obtained immediately by using Theorem 1, (b). Let us note some other immediate facts about separating functions.

<u>Lemma 1</u>. If $f : X \to Y$ is separating, then $f(X)$ is a Hausdorff subspace of Y. □

<u>Lemma 2</u>. If f is a continuous function from a space X into a Hausdorff space Y, then f is separating. □

<u>Lemma 3</u>. If $f : X \to Y$ is separating and K_1, K_2 are disjoint compact subspaces of Y, then there exist open sets U and V such that $K_1 \subset U$, $K_2 \subset V$ and $f^{-1}(U)$, $f^{-1}(V)$ are separated. □

<u>Lemma 4</u>. If $f : X \to Y$ is separating, $A \subset X$, $B \subset Y$ and $f(A) \subset B$, then $f|A : A \to B$ is separating. □

<u>Lemma 5</u>. If $f : X \to Y$ is separating, then any extension of the topology on either X or Y does not destroy the separativity of f. □

The last lemma permits the construction of separating functions that are not continuous. One can also easily obtain closed graph functions which are not separating. However we postpone giving an appropriate example until after our considerations concerning the dependence of separating functions on other classes of functions, e.g., Δ-closed graph functions or nearly continuous functions.

Following M. Wilhelm [7], we say that a function f from a space X into a space Y has a *Δ-closed graph* if the following condition is satisfied:

(Δ) if a net $\{(u_\sigma, v_\sigma)\}$ of points of $X \times X$ converges to the diagonal Δ_X and the net $\{(f(u_\sigma), f(v_\sigma))\}$ converges to a point (w,z), then $w = z$.

Let us recall that a net $\{a_\sigma\}$ converges to a set A if for any open set U containing A there is a σ_0 such that $x_\sigma \in U$ for all $\sigma \geq \sigma_0$.

<u>Lemma 6</u>. If $f : X \to Y$ has a Δ-closed graph, then it is separating.

<u>Proof</u>. Assume that the function f is not separating. There are different points $w, z \in Y$ such that $f^{-1}(W)$, $f^{-1}(Z)$ are not separated for each pair W, Z of open neighborhoods of points w, z, respectively. If so, then we may assume that $f^{-1}(W) \cap \mathrm{cl}\, f^{-1}(Z) \neq \emptyset$ for each pair W, Z of open neighborhoods of points w, z, respectively. Now we shall define a net (u_σ, v_σ) of points of $X \times X$. An index σ is an ordered triple (W,Z,D), where W is an open neighborhood of the point w, Z is an open neighborhood of the point z and D is an open neighborhood of the diagonal Δ_X. If $\sigma = (W,Z,D)$ and $\sigma' = (W', Z', D')$, then we set $\sigma \leq \sigma'$ iff $W' \subset W$, $Z' \subset Z$ and $D' \subset D$. For $\sigma = (W,Z,D)$ we choose (u_σ, v_σ) to be a point from $X \times X$ such that $f(u_\sigma) \in W$, $f(v_\sigma) \in Z$ and $(u_\sigma, v_\sigma) \in D$. This is possible for the following reason: since $f^{-1}(W) \cap \mathrm{cl}\, f^{-1}(Z) \neq \emptyset$ there is a u belonging to $f^{-1}(W) \cap \mathrm{cl}\, f^{-1}(Z)$. Let G be an open neighborhood of u such that $G \times G \subset D$. Since $u \in \mathrm{cl}\, f^{-1}(Z)$, $G \cap f^{-1}(Z) \neq \emptyset$. Hence there is a v belonging to $G \cap f^{-1}(Z)$. It suffices to put $u_\sigma = u$ and $v_\sigma = v$.

The net $\{(u_\sigma, v_\sigma)\}$ converges to the diagonal Δ_X and the net $\{(f(u_\sigma), f(v_\sigma)\}$ converges to the point (w,z).

Indeed, if D_0 is an open neighborhood of Δ_X, then let $\sigma_0 = (Y,Y,D_0)$. Then for any $\sigma = (W,Z,D)$ such that $\sigma_0 \leq \sigma$ we have, in particular, $D \subset D_0$ and hence $(u_\sigma, v_\sigma) \in D \subset D_0$.

If H is an open neighborhood of the point (w,z), then there are open neighborhoods W_0 of w and Z_0 of z such that $(W_0 \times Z_0) \subset H$. Let $\sigma_0 = (W_0, Z_0, X \times X)$. Then for any $\sigma = (W,Z,D)$ such that $\sigma_0 \leq \sigma$ we have, in particular, $W \subset W_0$ and $Z \subset Z_0$ and hence $(f(u_\sigma), f(v_\sigma)) \in W \times Z \subset W_0 \times Z_0 \subset H$.

Because $w \neq z$, the condition (Δ) is not satisfied and hence f does not have the Δ-closed graph. □

Example. There exists a separating function from a completely regular space onto a 0-dimensional regular paracompact space Y that does not have a Δ-closed graph.

Let X be the topological space with underlying set $(-1,1) \times (0,1) \cup (-1,0] \times \{1\} \cup [0,1) \times \{0\}$ and the topology defined in the following way:

Any point (x,y) such that $x \neq 0$ has the base of neighborhoods inherited from the product topology for $[-1,1] \times [0,1]$. Any point $(0,y)$ such that $0 \neq y \neq 1$ has the base of neighborhoods consisting of sets of the form $K^- \cup \{(0,y)\} \cup K^+$, where K^- is the set of all points of $(-1,0] \times (0,1)$ inside a circle that is tangent to $\{0\} \times (0,1)$ at y, and K^+ is the set of all points of $[0,1) \times (0,1)$ inside a circle that is tangent to $\{0\} \times (0,1)$ at y. Finally, the points $(0,1)$ and $(0,0)$ have the base of neighborhoods consisting of sets $K^- \cup \{(0,1)\}$ and $\{(0,0)\} \cup K^+$, respectively. Clearly X is completely regular (compare with the proof that Niemytzki's plane is completely regular [5]).

The space Y is formed from X by collapsing the sets $(-1,0] \times \{1\}$ and $[0,1) \times \{0\}$ to distinct points C_1, C_0 respectively.

The topology on Y is a base in the following way: each non-collapsed point is isolated; a base of neighborhoods for C_1 consists of sets of the form $\{C_1\} \cup ((-1,0) \times (0,1) - (\mathrm{cl}\, K_1^- \cup \ldots \cup \mathrm{cl}\, K_n^-))$, where $\mathrm{cl}\, K^-$ is a closed circle of the type K^-; similarly, a base of neighborhoods for C_0

consists of sets of the form $\{C_0\} \cup ((0,1) \times (0,1) - (\mathrm{cl}\ K_0^+ \cup \dots \cup \mathrm{cl}\ K_n^+))$. Observe that Y satisfies "good" separation axioms: it is for example 0-dimensional and paracompact.

Now, let f be the quotient function from X onto Y. We shall show that this function is separating but it does not have a Δ-closed graph.

In order to see that f is separating it is enough to check that the two "suspect" points C_1, C_0 have separated preimages under f of their neighborhoods $\{C_1\} \cup (-1,0) \times (0,1)$ and $\{C_0\} \cup (0,1) \times (0,1)$. So we pass to the proof that f does not have a Δ-closed graph. Let σ be an ordered triple (W,Z,D) where W is an open neighborhood of C_1, Z is an open neighborhood of C_0 and D is an open neighborhood of the diagonal Δ_x. As usual, the order for σ's is the inclusion between the corresponding members of the triples. Now for any such σ we shall define an ordered pair (u_σ, v_σ). Namely, if $\sigma = (W,Z,D)$ and $G = \{C_1\} \cup ((-1,0) \times (0,1) - (\mathrm{cl}\ K_1^- \cup \dots \cup \mathrm{cl}\ K_n^-))$ and $H = \{C_0\} \cup ((-1,0) \times (0,1) - (\mathrm{cl}\ K_1^+ \cup \dots \cup \mathrm{cl}\ K_m^+$ are basic neighborhoods of C_1, C_0 contained in W, Z, respectively, then let $(0,y)$ be any point of $\{0\} \times (0,1)$ not belonging to the set $\mathrm{cl}\ K_1^- \cup \dots \cup \mathrm{cl}\ K_n^- \cup \mathrm{cl}\ K_1^+ \cup \dots \cup \mathrm{cl}\ K_m^+$. Let U be any open neighborhood of the point $(0,y)$ such that $U \times U \subset D$. Then $G \cap U \neq \emptyset \neq H \cap U$ and it is enough to put $u_\sigma \in G \cap U$ and $v_\sigma \in H \cap U$. Clearly, the net $\{(u_\sigma, v_\sigma)\}$ converges to the diagonal Δ_X and the net $\{(f(u_\sigma), f(v_\sigma)\}$ converges to the point (C_1, C_0). Since $C_1 \neq C_0$, f does not have a Δ-closed graph.□

<u>Lemma 7</u>. Let Y be a k-space If $f : X \to Y$ has a closed graph and preimages under f of compact subsets of Y are compact, then f is separating.

<u>Proof</u>. Let u,v be arbitrary different points of Y. Applying Corollary 1 (c) to the compact sets $\{u\}$, $f^{-1}(v)$ and $\{v\}$, $f^{-1}(u)$ we obtain

open neighborhoods U', V' of the points u and v. respectively, such that

$$\mathrm{cl}\ f^{-1}(U') \cap f^{-1}(v) = \emptyset = f^{-1}(u) \cap \mathrm{cl}\ f^{-1}(V') .$$

By virtue of Theorem 4 (a), f is a closed function. Hence $U = U' \cap (Y - f(\mathrm{cl}\ f^{-1}(V')))$ is an open neighborhood of u and $V = V' \cap (Y - f(\mathrm{cl}\ f^{-1}(U')))$ is an open neighborhood of v and the sets $f^{-1}(U)$, $f^{-1}(V)$ are separated. □

A function $f : X \to Y$ is said to be *nearly continuous* if $f^{-1}(V) \subset \mathrm{int\ cl}\ f^{-1}(V)$ for every open set $V \subset Y$.

Near continuity brings continuity to closed graph functions in the absence of compactness. For example, one can easily observe that any closed graph nearly continuous function into a regular locally compact space is continuous (compare with Corollary 3 (b) and Example). Moreover, T. Byczkowski and R. Pol [3] have shown that closed graph nearly continuous functions preserving compact sets under preimages are continuous whenever their range is a Čech-complete space; M. Wilhelm [7] came to the same conclusion assuming Δ-graph closedness and near continuity. We now have the following.

<u>Theorem 8</u>. If $f : X \to Y$ is a separating nearly continuous function into a locally Čech-complete space Y, then f is continuous.

<u>Proof</u>. Assume, to the contrary, that there exists a point $p \in X$ and an open neighborhood $\tilde{V}$ of $f(p)$ such that $f(\tilde{U}) - \tilde{V} \neq \emptyset$ whenever $\tilde{U}$ is an open neighborhood of p. Let G be an open neighborhood of $f(p)$ such that $\mathrm{cl}\ G \subset \tilde{V}$ and G is a Čech-complete subspace of Y. Since f is nearly continuous, the inverse image $f^{-1}(Y - \mathrm{cl}\ G)$ of the open set $Y - \mathrm{cl}\ G$ is contained in $\mathrm{int\ cl}\ f^{-1}(Y - \mathrm{cl}\ G)$ and hence it follows that $f^{-1}(Y - \mathrm{cl}\ G) \cap \mathrm{cl}\ f^{-1}(G) \neq \emptyset$. Let g be any point from $f^{-1}(Y - \mathrm{cl}\ G) \cap \mathrm{cl}\ f^{-1}(G)$. There exists an open neighborhood H of $f(g)$ such that $\mathrm{cl}\ H \subset Y - \mathrm{cl}\ G$, and H is a Čech-complete subspace of Y. Let $\{A_n\}$ and $\{B_n\}$ be countable families of open covers of G and H, respectively, which

satisfy the conditions for G and H to be Čech-complete. Now choose a set H_1 which is an open neighborhood of $f(g)$ such that $\mathrm{cl}\, H_1 \subset B$ for some $B \in \mathcal{B}_1$. Then $\mathrm{cl}\, f^{-1}(H_1)$ is a neighborhood of g, and therefore $\mathrm{cl}\, f^{-1}(H_1) \cap f^{-1}(G) \neq \emptyset$. Let γ be any point from $\mathrm{cl}\, f^{-1}(H_1) \cap f^{-1}(G)$. Now choose a set G_1 which is an open neighborhood of $f(\gamma)$ such that $\mathrm{cl}\, G_1 \subset A$ for some $A \in \mathcal{A}_1$. Then $\mathrm{cl}\, f^{-1}(G_1)$ is a neighborhood of γ, and therefore $f^{-1}(H_1) \cap \mathrm{cl}\, f^{-1}(G_1) \neq \emptyset$. Continuing this process, one can find two sequences $\{G_n\}$ and $\{H_n\}$ satisfying the following conditions:

(1) $\mathrm{cl}\, G_{n+1} \subset G_n$ and $\mathrm{cl}\, H_{n+1} \subset H_n$ for each n;

(2) $\mathrm{cl}\, G_n$ has diameter less than $\mathcal{A}_n$ and $\mathrm{cl}\, H_n$ has diameter less than $\mathcal{B}_n$, for each n;

(3) $f^{-1}(H_n) \cap \mathrm{cl}\, f^{-1}(G_n) \neq \emptyset$ for each n.

The first two conditions guarantee that the sets $K_1 = \bigcap\{\mathrm{cl}\, G_n : n = 1,2,\ldots\}$ and $K_2 = \bigcap\{\mathrm{cl}\, H_n : n = 1,2,\ldots\}$ are nonempty compact subsets of the space Y and that the families $\{G_n\}$ and $\{H_n\}$ constitute open bases around these sets, respectively. Since $K_1 \subset G$ and $K_2 \subset H \subset Y - \mathrm{cl}\, G$, the sets K_1, K_2 are disjoint. By virtue of Lemma 3, there are open sets U and V such that $K_1 \subset U$, $K_2 \subset V$ and $f^{-1}(U)$, $f^{-1}(V)$ are separated. Since $\{G_n\}$, $\{H_n\}$ are decreasing bases around K_1, K_2, respectively, there is an index K such that $G_K \subset U$ and $H_K \subset V$. But then $f^{-1}(H_K) \cap \mathrm{cl}\, f^{-1}(G_K) = \emptyset$, contradicting (3). □

Our result, the proof of which is based on an idea due to Byczkowski-Pol [3], constitutes a common generalization of the previously mentioned results by Byczkowski-Pol and Wilhelm (see Lemmas 6 and 7). It also allows us to answer a question posed by M. Wilhelm [8]: will a space Y necessarily be Čech-complete if every nearly continuous Δ-closed graph function from any Hausdorff space X into Y is continuous? The answer is "no". Any locally Čech-complete but not Čech-complete space gives a negative answer, by virtue of our Theorem 8, and such spaces are known to exist (see for example

[5; p. 297]). Note also that any non-continuous function into a Čech-complete space that is nearly continuous and has a closed graph is an example of a closed non-separating function. Such a function was constructed in [3].

A regular space X is said to be of *pointwise countable type* if, for every point $x \in X$, there exists a compact set $C \subset X$ such that $x \in C$, and there exists a countable base of open neighborhoods of C. It is known [5] that any Čech-complete space is a space of pointwise countable type and that those spaces are *k*-space.

<u>Theorem 9</u>. If $f : X \to Y$ is a nearly continuous separating function into a countably compact space of pointwise countable type, then f is continuous.

<u>Proof</u>. Assume, to the contrary, that there exists a point $p \in X$ and an open neighborhood $\tilde{V}$ of $f(p)$ such that $f(\tilde{U}) - \tilde{V} \neq \emptyset$ whenever $\tilde{U}$ is an open neighborhood of p. Let C be a compact subset of Y containing $f(p)$ such that there is a countable base, say $\{V_n\}$, of open neighborhoods of C. Without loss of generality we may assume that $C \subset \tilde{V}$ and that $\{\tilde{V}_n\}$ is a decreasing base. Since f is nearly continuous, the sets $\mathrm{cl}\ f^{-1}(V_n)$ are open neighborhoods of p. Hence $\mathrm{int\ cl}\ f^{-1}(V_n) \cap f^{-1}(Y-\tilde{V}) \neq \emptyset$ for each n. Let $x_n \in \mathrm{int\ cl}\ f^{-1}(V_n) \cap f^{-1}(Y-\tilde{V})$. Since Y is countably compact and $Y-\tilde{V}$ is a closed subset of Y, there is a point $g \in Y-\tilde{V}$, which is an accumulation point of the sequence $\{f(x_n)\}$. The compact sets $\{g\}$ and C are disjoint, and therefore there are open sets U and V such that $g \in U$, $C \subset V$ and the sets $f^{-1}(U)$, $f^{-1}(V)$ are separated. Let us take V_n such that $V_n \subset V$. Then $f^{-1}(U) \cap \mathrm{cl}\ f^{-1}(V_n) = \emptyset$. However the set $\mathrm{cl}\ f^{-1}(V_n)$ contains all but finitely many elements of the sequence $\{x_u\}$, and the set $f^{-1}(U)$ contains infinitely many elements of the sequence $\{x_u\}$. Hence $f^{-1}(U) \cap \mathrm{cl}\ f^{-1}(V_n) \neq \emptyset$, a contradiction. □

Corollary 10. Let $f: X \to Y$ be a separating function into a regular second countable space Y. If Y is also either locally Čech-complete or countably compact, then there exists a residual subspace Z of X such that $f|Z : Z \to Y$ is continuous.

Proof. Let $\mathcal{B}$ be a countable base in Y. For each $V \in \mathcal{B}$ let $F_V = \mathrm{cl}\, f^{-1}(V) - \mathrm{int}\,\mathrm{cl}\, f^{-1}(V)$. Each of the sets F_V is nowhere dense and closed. Therefore $Z = X - \bigcup \{F_V : V \in \mathcal{B}\}$ is a residual subspace of X. The function $f|Z : Z \to Y$ is then nearly continuous. It is also separating (see Lemma 4). Now, if Y is locally Čech-complete then, applying Theorem 8, we conclude that function $f|Z$ is continuous; if Y is countably compact we obtain the same conclusion using Theorem 9, because first countable regular spaces are of pointwise countable type. $\square$

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