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CONCERNING BLUMBERG'S THEOREM

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1. **History.** In 1922, Blumberg [2] showed that if X is a separable complete metric space and Y is the set of reals, then X has the following property:

(*) If f is a function defined on X into Y, then there exists a dense subset D of X such that the restriction f|D is continuous.

If X is a space and Y is the set of reals, and (*) holds, then X is called a *Blumberg space*; the set D is called a *Blumberg set* for f.

In 1960, Bradford and Goffman [3] showed that if X is metric, then X is a Blumberg space iff X is a Baire space (i.e., a space for which the Baire category theorem holds). White (1974) proved in [7] that if X is a Baire space having a σ-disjoint pseudo-base, then X is a Blumberg space. In 1976, Alas [1] improved White's result by showing that if X is a Baire space having a σ-disjoint pseudo-base and Y is a second countable Hausdorff space, then (*) holds. Moreover, she showed that the second countability of Y is essential ([1], Example, page 582). The question of whether the assumption that Y is Hausdorff is essential was not answered (oral communication from O. T. Alas).

2. **Preliminaries and notations.** In this paper we prove, using completely different ideas than those of Alas or White, that if X is a Baire space having a σ-disjoint pseudo-base and Y is a second countable space (not necessarily Hausdorff), then (*) holds (see Theorem 8). We also show that the class of topological Blumberg spaces coincides with the class of topological spaces satisfying (*) for every second countable space Y. Next, we investigate some properties of Blumberg spaces, e.g., topological operations (subspaces, products) and the invariance of Blumberg spaces under some types of functions. Some remarks on Blumberg sets are also given.

Now we recall the definitions of some notions used in our paper.

A family P of non-empty open sets is said to be a *pseudo-base* if every non-empty open set contains a member of $\mathcal{P}$; $\pi w(X) = \min\{|\mathcal{P}|: \mathcal{P}$ is a pseudo-base of $X\}$.

A family $\mathcal{R}$ is called *σ-disjoint* if $\mathcal{R} = \cup\{P_n: n < \omega\}$, where each P_n is a disjoint family.

Let Y be a space. A space X will be called a $B^+(Y)$*-space* if (*) holds. Thus, the $B^+(R)$-spaces, R being the set of reals, are, by the definition the *Blumberg spaces*.

The *Novák number* of a dense-in-itself space X, $n(X)$, is the minimal cardinality of a family consisting of nowhere dense sets covering an open non-empty subset of X. Thus, for any dense-in-itself Baire space X, $n(X) > \omega_1$.

A function $f: X \to Y$ is called *perfect* if f is continuous and closed and, for each $y \in Y$, the preimage $f^{-1}(y)$ is compact.

A function $f: X \to Y$ is called *irreducible* if f is a continuous surjection and $\mathrm{cl}\, f(F) \neq Y$ for each closed proper $F \subset X$.

3. **An extension of Blumberg's theorem.** First we will show that the Blumberg spaces are $B^+(SC)$-spaces for each second countable space, abbreviated as SC.

THEOREM 1. *A space X is a Blumberg space iff, for every countable cover* $\mathcal{P}$ *of X, there exists a dense subset D of X such that* $P \cap D$ *is open in D for every* $P \in \mathcal{P}$.

PROOF. Assume that X is a Blumberg space. Let $\mathcal{P}$ be an arbitrary countable cover of X. For $P \in \mathcal{P}$, let χ_P be the characteristic function for P. Due to the countability of $\mathcal{P}$, the diagonal χ of the functions χ_P is a function from X into the Cantor set $\{0,1\}^\omega$. Since X is a Blumberg space, there is a dense set $D \subset X$ such that $\chi|D$ is continuous. But $\chi_P|D = (\pi_P \circ \chi)|D$, therefore also $\chi_P|D$ is continuous, for each $P \in \mathcal{P}$. This implies that $P \cap D$ is open in D.

Conversely, let f be an arbitrary real-valued function from X. Without loss of generality we may assume that f takes values in the interval $[0,1]$. Let h be a continuous function from $\{0,1\}^\omega$ onto $[0,1]$ and g be a function such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "g"'] & {[0,1]} \\ \{0,1\}^\omega \arrow[ur, "h"'] & \end{tikzcd}$$

commutes. Since h is onto, such g always exists. Let $\mathcal{P}$ be the cover of X consisting of the sets $g^{-1}(\pi_n^{-1}(i))$, where π_n denotes the natural projection from $\{0,1\}^\omega$ onto the

n-th axis, $i \in \{0,1\}$ and $n < \omega$. Evidently, P is countable, so there is a dense set $D \subset X$ with $P \cap D$ being open in D, for $P \in \mathcal{P}$. Hence $(\pi_n \cdot g)|D$ is a continuous function for every $n < \omega$. Therefore $g|D$ is continuous. We conclude that $f|D = (h \cdot g)|D = (h|D) \cdot (g|D)$ is continuous.

COROLLARY 2. *A space X is a Blumberg space iff it is a* $B^+(SC)$-*space for each SC.*

PROOF. Let X be a Blumberg space, let Y be an arbitrary second countable space having a countable base $\mathcal{B}$, and let $f: X \to Y$ be a function. Then $\mathcal{P} = \{f^{-1}(B): B \in \mathcal{B}\}$ is a countable cover of X. It follows from Theorem 1 that there is a dense subset D of X with $P \cap D$ being open in D, for each $P \in \mathcal{P}$. Thus $f|D$ is continuous.

The converse implication is obvious.

LEMMA 3. *Let f be a continuous function from X onto Y. The following conditions are equivalent:*

(1) *f is irreducible:*

(2) $f^{-1}(\mathcal{P}) = \{f^{-1}(P): P \in \mathcal{P}\}$ *is a pseudo-base of X, for each pseudo-base* $\mathcal{P}$ *of the space Y;*

(3) *there exists a pseudo-base* $\mathcal{P}_0$ *of the space Y such that* $f^{-1}(\mathcal{P}_0)$ *is a pseudo-base of X.*

PROOF. (1) $\Rightarrow$ (2). Let $\mathcal{P}$ be a pseudo-base of Y and let U be an open non-empty subset of X. Since f is irreducible we have $\mathrm{cl}\, f(X - U) \neq Y$. Thus there is $P \in \mathcal{P}$ such that $P \subset Y - \mathrm{cl}\, f(X - U)$ from which $f^{-1}(P) \subset U$. Hence $f^{-1}(\mathcal{P})$ is a pseudo-base of X.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $\mathcal{P}_0$ be a pseudo-base of X, with $f^{-1}(\mathcal{P}_0)$ being a pseudo-base of X. If F is a closed proper subset of X, then there is a $P \in \mathcal{P}_0$ such that $f^{-1}(P) \cap F = \emptyset$. Therefore $f(F) \cap P = \emptyset$, hence $\mathrm{cl}\, f(F) \neq Y$.

LEMMA 4. *Let X be a Baire space. If* $f: X \to Y$ *is an irreducible surjection, then Y is a Baire space.*

LEMMA 5. *Let X be a Baire space having a σ-disjoint pseudo-base. Then X contains a dense* G_δ *set Z for which there is an irreducible surjection* $f: Z \to Y$, *where Y is a metric space.*

PROOF. Let $\mathcal{P}$ be a σ-disjoint pseudo-base of X. We may assume $\mathcal{P} =$

$\mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots$, where each $\mathcal{P}_i$ consists of disjoint sets which cover X densely. Put $Z = \bigcap_{i<\omega}(\cup\{P: P \in \mathcal{P}_i\})$. Hence Z is a dense G_δ set in X. Let $f_i: Z \to \mathcal{P}_i$ be the function such that $f_i(x) = P$ iff $x \in P \in \mathcal{P}_i$. Each of the functions f_i is continuous where $\mathcal{P}_i$ is endowed the discrete topology. The diagonal function f of the f_i's is also continuous. The function f takes values in the countable product of discrete spaces $\mathcal{P}_i$, a metric space. Consider $\mathcal{B} = \{\pi_i^{-1}(P): P \in \mathcal{P}_i; i < \omega\}$, observe that $f^{-1}(\mathcal{B})$ is a pseudo-base of Z, hence, by Lemma 3, the function f is irreducible.

LEMMA 6. *Let Y be a Blumberg space. If* $f: X \to Y$ *is an irreducible surjection, then X is a Blumberg space.*

PROOF. Let $\mathcal{P}$ be a countable cover of the space X. Then $f(\mathcal{P}) = \{f(P): P \in \mathcal{P}\}$ is a countable cover of Y. Since Y is a Blumberg space, there is a dense set $D' \subset Y$ such that $f(P) \cap D'$ is open in D' for each $P \in \mathcal{P}$. In each set of the form $f^{-1}(y)$, y belonging to D', choose precisely one element and let D be the set consisting of these elements. Then $f|D$ is injective and $f(D) = D'$. Since D' is dense in Y, the set D is dense in X, f being irreducible. So, if $P \in \mathcal{P}$, then $P \cap D = f^{-1}(f(P)) \cap D$. Now, $f|D$ is continuous and thus $P \cap D$ is open in D. Hence, by Theorem 1, X is a Blumberg space.

Recall the following characterization due to Bradford and Goffman.

THEOREM 7. ([3]) *Let X be a metric space. Then X is a Blumberg space iff X is a Baire space.*

Using the just quoted result of Bradford and Goffman we will prove the following theorem.

THEOREM 8. *If X is a Baire space having a σ-disjoint pseudo-base, then X is a* $B^+(SC)$*-space.*

PROOF. It follows immediately from Lemmas 5 and 6, Theorem 7, and Corollary 2.

4. Blumberg spaces: topological operations and a mapping approach. It is easy to see that a dense subspace of a Blumberg space need not be a Blumberg space. Recall that the Stone-Čech compactification of a dense subspace of a completely regular Blumberg space is a Blumberg space ([5], Theorem 4.1).

The Cartesian product of Blumberg spaces need not be a Blumberg space, since Fleissner and Kunen [4] constructed a metric Baire space (thus a Blumberg space) dense subset of X such that $D \cup \{x\}$ is a Blumberg set for each $x \in X$. Let $x_0 \in X$ and let $y_0 = f(x_0)$.

Suppose there is an open set G such that $y_0 \in G$ and $U \cap D \cap f^{-1}(Y \setminus cl\, G) \neq \emptyset$ for each open set U with $x_0 \in U$. Then $D \cap f^{-1}(Y \setminus cl\, G)$ has x_0 as a limit point because $x_0 \notin f^{-1}(Y \setminus cl\, G)$. Moreover, the fact that $D \cup \{x_0\}$ is a Blumberg set implies $f(x_0) \notin G$ because G is open. That is, $y_0 \notin G$ and $y_0 \in G$, a contradiction. Consequently, we have that, for each open set G with $y_0 \in G$, there is an open set U with $x_0 \in U$ such that

$$U \cap D \cap f^{-1}(Y \setminus cl\, G) = \emptyset.$$

Since D is dense in X and $D \cup \{x\}$ is a Blumberg set for each $x \in U$, we conclude that

$$f(U) \subset cl\, f(U \cap D) \subset cl\, G.$$

Because Y is a regular space, f is continuous at x_0. Since x_0 is an arbitrary point of X, we have that f is continuous.

COROLLARY 12. *Let Y be a regular space and let f: $X \to Y$ possess a Blumberg set. Then there is a maximal Blumberg set for f.*

PROOF. Let $\mathcal{D} = \{D_\lambda | \lambda \in \Lambda\}$ be a chain of Blumberg sets of f. Let λ_0 be fixed and let $X_0 = \cup_{\lambda \in \Lambda} D_\lambda$. Clearly, D_{λ_0} is dense in X_0. Let $x \in X_0$. Since $\mathcal{D}$ is a chain, there is a λ so that $x \in D_\lambda$ and $D_\lambda \supset D_{\lambda_0}$. Because D_λ is a Blumberg set for f and D_{λ_0} is dense in X_0, $D_{\lambda_0} \cup \{x\}$ is a Blumberg set for $f|X_0$. The theorem now implies $f|X_0$ is continuous. Also, X_0 is dense in X. So, X_0 is a Blumberg set for f. The proof of Corollary 12 now follows by the Kuratowski-Zorn lemma.

The assumption of regularity of the space Y in Theorem 11 cannot be omitted.In fact, we have the following example.

EXAMPLE 13. Consider the topology $\mathcal{T}$ for the reals, generated by:

$$U(X,n) = (x - \frac{1}{n}, x + \frac{1}{n}), \text{ for } x \in Q,$$

$$U(x,n) = (x - \frac{1}{n}, x + \frac{1}{n}) \setminus \{x \pm \frac{x}{m} | m = 1,2,...\}, \text{ for } x \in R \setminus Q.$$

The topology $\mathcal{T}$ is Hausdorff, but not regular. Let f: $(R,E) \to (R,\mathcal{T})$ be a function defined by $f(x) = x$, for each $x \in R$, E being the Euclidean topology for the reals. It is easy to see that Q is the set of points of continuity of f. For every $x \in R$, the set

$Q \cup \{x\}$ is the Blumberg set for f, however the function f is not continuous.

It is easy to find a metric space X and a function $f: X \to R$ such that there is no minimal (with respect to inclusion) Blumberg set for f.

REMARK 14. Corollary 2 gives a positive answer to H. E. White's question (see [8], page 468) whether every Blumberg space is a B^+(SC)-space. The authors thank Professor J. B. Brown for calling attention to this paper.

REMARK 15. Some results from the first part of this paper have been announced in Proceedings of the Conference on Topology and Measure II, Griefswald, GDR, 1980.

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