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A CURVE IN R^n, HOMEOMORPHIC TO A LINE, THAT INTERSECTS EVERY UNBOUNDED CURVE OF BOUNDED CURVATURE

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A. V. KUZ'MINYKH

As is known (see for example [1], pp. 222–223), there exists a simple arc (i.e., a set homeomorphic to the unit interval) in R^3 whose projection (everywhere here a projection means the orthogonal projection) on the plane $z = 0$ is a square (this simple arc is obtained by, so to speak, unwinding the Peano curve).

The question naturally arises whether there exists a curve in R^n which is homeomorphic to a line (i.e., a subset of R^n which is homeomorphic to a (straight) line, or which is the same as the image of a straight line under its embedding in R^n; such a curve is also called *elementary*), whose projection on every hyperplane is the entire hyperplane (that is, this curve intersects every line $L \subset R^n$).

In a more general manner, the question comes up: How "enlarged" can the class of curves intersecting a curve which is homeomorphic to a line become? Theorem 1, formulated below, gives us an answer to this question.

A *curve* is defined as the image of a (straight) line under a continuous mapping into R^n, $n \geq 2$.

We will call a simple arc $\Gamma \subset R^n$ *Lipschitz* if there exist a cartesian rectangular orthonormal coordinate system (in R^n) and positive numbers M and a such that Γ can be represented in this coordinate system by the equations $x_1(t) = t$, $x_2(t), \dots, x_n(t)$, $t \in [0, a]$, where the functions $x_i(t)$, $i = 2, \dots, n$, are such that for every $t_1, t_2 \in [0, a]$

$$\left(\sum_{i=2}^{n}(x_i(t_1) - x_i(t_2))^2\right)^{1/2} \leq M|t_1 - t_2|$$

(such a simple arc we will also call *M-Lipschitz*; we remark that the functions $x_i(t)$, generally speaking, are not assumed to be differentiable).

We will call a curve $\gamma \subset R^n$ *piecewise Lipschitz* if it is the union of countably many Lipschitz simple arcs.

We will call a curve $\gamma \subset R^n$ *admissible* if there is an unbounded set $\mathscr{M}_\gamma \subset R^n$ such that $\rho(\gamma, \mathscr{M}_\gamma) > 0$, where ρ is the Euclidean metric in R^n. (Thus, for example, a curve γ is admissible if there is an unbounded convex body $V_\gamma \subset R^n$ for γ such that $\gamma \cap V_\gamma = \varnothing$.)

We will call a curve $\gamma \subset R^n$ *a curve of bounded curvature* if γ is a regular (twice continuously differentiable) elementary curve for which there is a number $M(\gamma) > 0$ such that the curvature of γ at every point is less then $M(\gamma)$.

If $\Gamma \subset R^n$ is a rectifiable arc, then (since Γ with the intrinsic metric defined on it is isometric to a rectilinear sgement) there can be defined a (linear) Lebesgue measure m_Γ on Γ (as well as on this segment) such that $m_\Gamma(\Gamma)$ equals the length of Γ.

Analogously, a measure m_γ can be defined on a curve γ of bounded curvature so that, in particular, for every simple arc $\Gamma \subset \gamma$, $m_\gamma(\Gamma)$ equals the length of the arc Γ).

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Therefore, the moving frame $\{\mathbf{e}_i\}$ of a Monge manifold of circular cones can be defined in terms of the given $\{\mathbf{m}_i\}$, the function α, and a certain parameter φ—the angle between the vector $\mathbf{m}_1$ and the projection of $\mathbf{e}_1$ on the corresponding plane of the Pfaffian manifold.

Thus, the invariants of a Monge manifold of circular cones can be determined in terms of the invariants of a Pfaffian manifold, the function α, and the parameter φ.

If we use the notation $\theta_3^2 = a_i\theta^i$ and $\theta_3^1 = b_i\theta^i$, then we can get expressions for the coefficients p and q in terms of the coefficients a_i and b_i, the parameter φ, and the function α.

In the proof of Theorem 1 in this case we obtain the system of two equations

$$q_{44} = 0, \qquad q_1 - p_2 = 0$$

after the appropriate transformations. The first equation means that C_3^2 is a Monge manifold of circular cones, and the second equation, after we write q_1 and p_2 in terms of a_i, b_i, α, and φ, gives us that (a) $Z = 0$, where Z is the nonholonomicity scalar of the Pfaffian manifold, and (b) $\alpha_1 = b_3 \cot\alpha$ and $\alpha_2 = a_3 \cot\alpha$. Theorem 1 follows from (a), and Theorem 2 from (b).

Theorem 3 is proved with the help of the following lemma:

Lemma. *A Monge manifold of circular cones is given by a Monge equation of the form* $(\theta^1)^2 + (\theta^2)^2 - (\theta^3)^2\tan^2\alpha = 0$, *where the* θ^i *are the principal forms of the Pfaffian manifold.*

Tomsk State University

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In what follows, card $\mathscr{M}$ will denote the cardinality of the set $\mathscr{M}$, $\mathfrak{c}$ the cardinality of the continuum, N the set of natural numbers, R the set of real numbers, and $\mathscr{K}$ the Cantor perfect set, constructed in the usual manner on $[0, 1]$.

The open interval $(0, 1) \subset R$ will be denoted by J. A function $f: J \to R^n$ will be called an *isometry at a point* $x \in J$ if there is an open interval J_x such that $x \in J_x \subset J$ and the restriction of f to J_x is the isometry (thus, in particular, $f(J_x)$ is a rectilinear interval).

THEOREM 1. *There is a curve $\mathfrak{M} \subset R^n$, $n \geq 2$, which is homeomorphic to a (straight) line and has the following properties:*

1) *For every hyperplane $P \subset R^n$, the projection of $\mathfrak{M}$ on P is the entire hyperplane P.*

2) *For every number $\alpha \in R$ there is a set $\mathfrak{N}_\alpha \subset \mathfrak{M}$, homeomorphic to $\mathscr{K}\backslash\{0\}$, such that, for every unbounded admissible piecewise Lipschitz curve $\gamma \subset R^n$,*

$$\operatorname{card}(\gamma \cap \mathfrak{N}_\alpha) = \mathfrak{c},$$

whereas if $\alpha_1 \neq \alpha_2$, $\alpha_1, \alpha_2 \in R$, then $\mathfrak{N}_{\alpha_1} \cap \mathfrak{N}_{\alpha_2} = \varnothing$.

3) *For every number $i \in N$ there is a set $\mathfrak{S}_i \subset \mathfrak{M}$ which is homeomorphic to $\mathscr{K}\backslash\{0\}$ and such that for every unbounded admissible piecewise Lipschitz curve $\gamma \subset R^n$ there is a Lipschitz simple arc $\Gamma \subset \gamma$ for which*

$$m_\Gamma(\Gamma \cap \mathfrak{M}) \geq m_\Gamma(\Gamma \cap \mathfrak{S}_i) > 0,$$

whereas if $i_1 \neq i_2$, $i_1, i_2 \in N$, then $\mathfrak{S}_{i_1} \cap \mathfrak{S}_{i_2} = \varnothing$.

4) *For every number $i \in N$ there is a set $\mathfrak{T}_i \subset \mathfrak{M}$ which is homeomorphic to $\mathscr{K}\backslash\{0\}$ and such that for every unbounded curve $\gamma \subset R^n$ of bounded curvature*

$$m_\gamma(\gamma \cap \mathfrak{M}) = m_\gamma(\gamma \cap \mathfrak{T}_i) = \infty,$$

whereas if $i_1 \neq i_2$, $i_1, i_2 \in N$, then $\mathfrak{T}_{i_1} \cap \mathfrak{T}_{i_2} = \varnothing$.

5) *There is a set $\mathfrak{U} \subset \mathfrak{M}$ homeomorphic to $\mathscr{K}\backslash\{0\}$ and such that for every plane $P \subset R^n$, $1 \leq \dim p \leq n$,*

$$m_P(P\backslash\mathfrak{M}) \leq m_P(P\backslash\mathfrak{U}) < 1,$$

where m_p denotes the standard ($\dim P$*-dimensional) Lebesgue measure in P.*

6) *There is a homeomorphic $\varphi: J \to \mathfrak{M}$ which is the isometry almost everywhere (i.e., there is a set $\Omega \subset J$ of linear measure 0 so that for every point $x \in J\backslash\Omega$ the function φ is the isometry at x).*

COROLLARY 1. *There is a curve $\mathfrak{M} \subset R^n$, $n \geq 2$, which is homeomorphic to a (straight) line and such that, for every regular (continuously differentiable) unbounded curve $\gamma \subset R^n$ whose projection on at least one hyperplane is not everywhere dense in this hyperplane,*

$$\operatorname{card}(\gamma \cap \mathfrak{M}) = \mathfrak{c}.$$

REMARK 1. It follows from property 5) of the curve $\mathfrak{M}$ in Theorem 1 that $m_n(\mathfrak{M}) = \infty$ (where m_n is (n-dimensional) Lebesgue measure in R^n). There is, however, a curve $\mathfrak{M}_0 \subset R^n$, $n \geq 2$, which is homeomorphic to a straight line and such that $m_n(\mathfrak{M}_0) = 0$ and $\mathfrak{M}_0$ satisfies 2) and consequently 1) (with $\mathfrak{M}$ replaced by $\mathfrak{M}_0$, of course) and also 6).

Additionally, for every $\varepsilon > 0$ there is a curve $\mathfrak{M}_\varepsilon \subset R^n$, $n \geq 2$, which is homeomorphic to a line and such that $m_n(\mathfrak{M}_\varepsilon) < \varepsilon$ and the curve $\mathfrak{M}_\varepsilon$ possesses properties

1)–3), 6) and also the following ones:

4′) For every number $i \in N$ there is a set $\mathfrak{T}_i^c \subset \mathfrak{M}_c$ which is homeomorphic to $\mathscr{K}\backslash\{0\}$ and such that for every unbounded curve $\gamma \subset R^n$ of bounded curvature

$$m_\gamma(\gamma \cap \mathfrak{M}_c) \geq m_\gamma(\gamma \cap \mathfrak{T}_i^c) > 0,$$

whereas if $i_1 \neq i_2$, $i_1, i_2 \in N$, then $\mathfrak{T}_{i_1}^c \cap \mathfrak{T}_{i_2}^c = \varnothing$.

5′) There is a set $\mathfrak{U}_c \subset \mathfrak{M}_c$ which is homeomorphic to $\mathscr{K}\backslash\{0\}$ and such that for every plane $P \subset R^n$, $1 \leq \dim P \leq n$,

$$m_P(P \cap \mathfrak{M}_c) \geq m_P(P \cap \mathfrak{U}_c) > 0.$$

REMARK 2. It is easy to see that property 1) of the curve $\mathfrak{M}$ (see Theorem 1) is invariant under "sufficiently nice deformations" of $\mathfrak{M}$. More precisely, we shall call a homeomorphism $\mathscr{F}: R^n \to R^n$ *canonical* if for every straight line $l \subset R^n$, $\mathscr{F}^{-1}(l)$ is a piecewise Lipschitz unbounded admissible curve. The curve $\mathfrak{M}$ has the following property: For every canonical homeomorphism $\mathscr{F}: R^n \to R^n$ and for every hyperplane $P \subset R^n$ the projection of $\mathscr{F}(\mathfrak{M})$ on P is the entire hyperplane P.

The closed disk and sphere (in R^n) centered at x having radius r will be denoted by $D^n(x, r)$ and $S^{n-1}(x, r)$, respectively.

Given positive numbers r_1 and r_2, $r_1 < r_2$, the open layer (in R^n) between concentric (with the center at the origin O) spheres with radii r_1 and r_2 will be denoted by $V_n(r_1, r_2)$, i.e.,

$$V_n(r_1, r_2) \overset{\text{def}}{=} (\operatorname{int} D^n(O, r_2))\backslash D^n(O, r_1).$$

If $\delta \subset R^n$ is a simple arc or a curve of bounded curvature, we shall say that δ *intersects a layer* $V_n(r_1, r_2)$ *canonically* if for each $i = 1, 2$, $\delta \cap S^{n-1}(O, r_i) \neq \varnothing$.

Let us recall that a zero-dimensional perfect compact set is called a *discontinuum*. As we know, all discontinua are homeomorphic to the Cantor set $\mathscr{K}$. A compact set $A \subset R^n$ is called (see [2]) *cellwise partitioned* in R^n if in each neighborhood V of it one can inscribe a neighborhood U whose closure cl U is the union of finitely many pairwise disjoint n-dimensional topological disks, cl $U \subset V$.

THEOREM 2. *Let ε, M, r_1, and r_2 be positive numbers, $r_1 < r_2$, and $\mathscr{N}$ a natural number. Then there exists a cellwise partitioned in R^n, $n \geq 2$, discontinuum $\mathfrak{D} \subset V_n(r_1, r_2)$ having the following properties:*

1) For every number $\alpha \in R$ there is a discontinuum $\mathfrak{C}_\alpha \subset \mathfrak{D}$ such that for every simple arc $\Gamma \subset R^n$ that canonically intersects $V_n(r_1, r_2)$ and is the union of $\mathscr{N}$ simple arcs, each being M-Lipschitz,

$$\operatorname{card}(\Gamma \cap \mathfrak{C}_\alpha) = c,$$

whereas if $\alpha_1 \neq \alpha_2$, $\alpha_1, \alpha_2 \in R$, then $\mathfrak{C}_{\alpha_1} \cap \mathfrak{C}_{\alpha_2} = \varnothing$.

2) For every simple arc $\Gamma \subset R^n$ that canonically intersects $V_n(r_1, r_2)$ and is the union of $\mathscr{N}$ simple arcs, each of them M-Lipschitz,

$$m_\Gamma(\Gamma \cap \mathfrak{D}) > 0.$$

3) For every curve $\gamma \subset R^n$ of bounded curvature whose curvature at every point is less than M, and which canonically intersects $V_n(r_1, r_2)$,

$$m_\gamma(\gamma \cap \mathfrak{D}) > r_2 - r_1 - \varepsilon.$$

4) For every plane $P \subset R^n$, $1 \leq \dim P \leq n$,

$$m_P(P \cap \mathfrak{D}) > m_P(P \cap V_n(r_1, r_2)) - \varepsilon.$$

COROLLARY 2. *If a discontinuum* $\mathfrak{D}$ *is cellwise partitioned in* R^n, *then it follows from Theorem* 1 (*see* [2]) *that there exists a simple arc* $\Gamma(\mathfrak{D})$ *such that* $\mathfrak{D} \subset \Gamma(\mathfrak{D}) \subset V_n(r_1, r_2)$ *and there is an isotopy* $F_t: R^n \to R^n$, $0 \leq t \leq 1$, *starting from the identity function* F_0 *such that* $F_1(\Gamma(\mathfrak{D}))$ *is a line segment.*

COROLLARY 3. *Let* r_1 *and* r_2 *be positive numbers,* $r_1 < r_2$. *Then there is a curve* $\mathfrak{L} \subset V_n(r_1, r_2)$, $n \geq 2$, *which is homeomorphic to a straight line and such that for every hyperplane* $P \subset R^n$, *the projection of* $\mathfrak{L}$ *on* P *is an open* (*in* P) $(n-1)$*-dimensional disk having radius* r_2 (*centered at the projection image of the point* O).

Institute of Mathematics
Siberian Branch, Academy of Sciences of the USSR
Novosibirsk
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