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THE GENESIS OF SEPARATE VERSUS JOINT CONTINUITY

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ABSTRACT. A historical account of the origins of the separate and joint continuity is presented.

1. Introduction

Let X, Y and Z be spaces, say $X = Y = Z = \mathbb{R}$, and let $f: X \times Y \to Z$ be a function. For every fixed $x \in X$, the function $f_x: Y \to Z$ defined by $f_x(y) = f(x, y)$ where $y \in Y$, is called an *x-section* of f. A *y*-section f_y is defined similarly. We say that a function f is *separately continuous* if all of its x-sections f_x and y-sections f_y are continuous.

When you ask a mathematician about the relationship between separate continuity and (joint) continuity you are likely to hear:

> "Separate continuity of a function $f: \mathbb{R}^2 \to \mathbb{R}$ does not imply the continuity of f. In fact, an example of such a function is given by:[1]
>
> $$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

An analyst may add:[2]

> "It follows from a theorem of R. Baire, that a separately continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ is of the first class of Baire. As such, it is pointwise discontinuous, i.e., the set $D(f)$ of points of discontinuity of f is of first category. Now, since the Baire Category Theorem

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[1] We shall refer to this function as Peano's example [GP]

[2] Professor W. Rudin has actually asked this question to some of his colleagues, (see [R1])

holds for the plane, such an f has a dense, G_δ set $C(f)$ of points of continuity."

In this article we organize known results, challenging, sometimes well established "facts" in this area starting from the early years of real analysis, going back to the years of A. L. Cauchy.

Regarding historical inaccuracies, Professor W. Rudin ([R3] p. 411) remarks that A. Tychonoff proved that arbitrary products of intervals are compact and used it to construct what is now known as the Čech (or Stone-Čech) compactification of a completely regular space. E. Čech proved the general case of the theorem and studied properties of the compactification. So, "... it appears that Čech proved the Tychonoff theorem, whereas Tychonoff found the Čech compactification — a good illustration of the historical reliability of mathematical nomenclature ... "

Apparently, a similar situation takes place when we go to the roots of separate versus joint continuity.

In this article we show that the first example of a separately continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that is not continuous — generally referred to as J. Thomae's example — is really due to E. Heine. Also, much simplified in form, the example given 14 years later by G. Peano, labeled as A. Genocchi's example, has its roots in Heine's construction, too.

In the conclusion we list the highlights of the early discoveries in this area in the beginning of the twentieth century.

One more thing; some of the first researchers in this area were rather interested in a closely related problem of the existence of the limit $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ of the function f when the limit of both of the sections f_x and f_y at the point (x_0, y_0) exists. This is why in some of the papers the function in question is not defined at $(0,0)$. Since only a little step is needed, in this article we will naturally extend the original function where necessary.

2. J. Thomae's example disproving A. Cauchy's "theorem" on separate continuity is due to E. Heine

Going back to the Cro-Magnon days of modern analysis, it was Jean Bernoulli in 1718 who gave the definition of a function (see also [LD]). The year 1821 brought "Cours d'Analyse" by Augustin-Louis Cauchy [Ca], and with this the installation of rigor in analysis. However, as M. Kline remarks ([K2] p. 176), although the "Cours" is "... essentially correct..., the language he (A. Cauchy) used was nevertheless vague and imprecise ... ". Following the definition of an infinitely small quantity ("quantité infiniment petite" ([Ca]

p. 19)), Cauchy proceeds with the definition of continuity ([Ca] p. 43; see also [K1] p. 952) as follows:

"... We also say that the function $f(x)$ is a continuous function of x in the neighborhood of a particular value assigned to the variable x, as long as it (the function) is continuous between those two limits of x, no matter how close together, which enclose the value in question ... "

A. Cauchy believed that if a function of two variables possesses a limit at some point, when each variable separately approaches that point, the function must approach a limit when both variables vary simultaneously and approach the point (see [Ca]; compare also [K2] p. 76).

Further, Cauchy in his "Cours" asserted that if a function of several variables is continuous in each one separately, it is a continuous function of all the variables. This is not correct (!) (see [K1] p. 952).

Like his contemporaries, he also believed that continuity implied differentiability (?!) (see Chapter VII of [Ca]), and even after his attention was called to his mistake, he ... persisted (compare [K2] p. 176).

As P. Lelong and P. Dugac [LD] and A. Rosenthal [Ro] indicate, it was J. Thomae [T1], "Abriss einer Theorie der complexen Funktionen," Halle, 1870 p. 15 who is *the first* to exhibit a counterexample to Cauchy's "theorem." We will challenge this statement.

So, let us *compare* both editions of Thomae's "Abriss."[3] In the first edition of this text, in 1870, J. Thomae mentions the name of E. Heine in the footnote on p. 13, *two pages* before the famous example is given, in connection with a discussion on complex variables.

Then two pages later, p. 15, in the footnote (which constitutes the majority of the page), with no reference to E. Heine whatsoever, J. Thomae states:

"... Die Funktion $\sin(4\operatorname{arctg}\frac{y}{z})$ ist, wie man sich leicht überzeugt, eine stetige Funktion von y und z in der Umgebung des Punktes $y = 0$, $z = 0$, wenn dieser Punkt selbst ausgeschieden wird ... "

"... The function $\sin(4\operatorname{arctg}\frac{y}{z})$ is, as one can easily verify, a continuous function of y and z in the neighborhood of the point $y = 0$, $z = 0$, if this point itself is excluded. ... ."

The second (1873) edition of Thomae's "Abriss" drastically differs from its 1870 version. E. Heine is not mentioned on p. 13; however, on p. 15 of [T2] we read:

[3] A. Rosenthal quotes, the second edition of Thomae's "Abriss" of 1873.

COURS D'ANALYSE

DE

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FIGURE 1

"... Man verfällt leicht in den Fehler (worauf Herr E. Heine) aufmerksam gemacht hat) in einem Gebiete eine Funktion zweier Veränderlichen für stetig zu halten, wenn in jedem Punkte $\mathrm{abs}[\omega(y \pm \zeta\delta, z) - \omega(y,z)]$ und $\mathrm{abs}[\omega(y, z \pm \zeta'\delta) - \omega(y,z)]$ mit abnehmendem δ gegen Null konvergieren. Dann müßte z. B. die Funktion $\omega(y,z) = \sin 4 \operatorname{arctg} \frac{y}{z}$, welche wir für $z = 0$ dadurch definieren, daß wir sie längs der ganzen y Achse (in der y, z-Ebene) gleich Null annehmen, im Innern des Kreises $y^2 + z^2 = 1$ überall stetig sein ..."

"One can easily commit the error (as Mr. E. Heine has pointed out) of considering a function of two variables to be continuous in an area if for every point $\mathrm{abs}[\omega(y \pm \zeta\delta, z) - \omega(y,z)]$ and $\mathrm{abs}[\omega(y, z \pm \zeta'\delta) - \omega(y,z)]$ converge toward zero with decreasing δ. However, that would mean, for example, that the function $\omega(y,z) = \sin 4 \operatorname{arctg} \frac{y}{z}$, which we define for $z = 0$ by assuming that it is equal to zero along the entire y axis (in the y, z plane), would be continuous every where within the circle $y^2 + z^2 = 1$."

So, E. Heine spotted A. Cauchy's error and communicated it to J. Thomae, who in turn, incorporated it into his textbook on complex variables!

3. Genocchi's, or rather Peano's example is essentially Heine's

Now, let us start with the function by E. Heine

$$F(y,z) = \begin{cases} \sin 4 \operatorname{arctg} \frac{y}{z}, & \text{if } z \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Let us introduce the cylindrical coordinate system

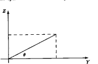

FIGURE 2

$$\sin\theta = \frac{z}{r} \quad \Longrightarrow \quad z = r\cdot\sin\theta\,,$$
$$\cos\theta = \frac{y}{r} \quad \Longrightarrow \quad y = r\cdot\cos\theta\,.$$

Let us work out the "$\sin 4\arctan\frac{y}{z}$" part:

$$= \sin\left(4\arctan\left(\tfrac{r\cos\theta}{r\sin\theta}\right)\right) = \sin 4\arctan(\cot\theta) = \sin 4\arctan\left[\tan\left(\tfrac{\pi}{2}-\theta\right)\right] =$$
$$= \sin(2\pi - 4\theta) = \sin\left[(-4\theta) + 2\pi\right] =$$
$$= \sin(-4\theta) = -\sin 4\theta\,.$$

So, H e i n e 's example in cylindrical coordinates is:

$$F(r,\theta) = \begin{cases} -\sin 4\theta\,, & \text{if } (r,\theta) \neq (0,0)\,, \\ 0, & \text{at } (0,0)\,. \end{cases}$$

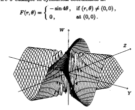

FIGURE 3

The essence of H e i n e 's example[4] lies in the fact that the graph of the function F is at the same level (here it equals 0) on the W-axis for all y's from a certain δ_1-neighborhood of 0 on the Y-axis and for all z's from a certain δ_2-neighborhood of 0 on the Z-axis. This condition follows from the separate continuity requirement on such a function.

Now, let us "take a journey" within the first quadrant on the graph of H e i n e 's F function. We shall be travelling counter-clockwise on the graph

[4] We take this opportunity to make the following correction in [Pi]. Replace "function continuous along every straight line through every point in its domain" by "separately continuous" on p. 294_{13}.

above the unit circle $r = 1$ (or any other "loop" around $(0,0)$). Starting from level 0 we feel the graph of F first dropping to level -1, then rising to level 1, and returning to the level 0 by the end of the trip. But this "roller coaster" is —mathematically speaking—not needed. Also, the sign "$-$" in the definition of F is obsolete.

If we are to stay in the framework of sine-like functions, in addition to the separate continuity condition illustrated above, it is sufficient that the graph of such a function starts from a "zero level" drops off/goes up and then goes up/drops off respectively to the "zero level" within at least one quadrant.

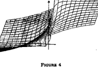

FIGURE 4

Observe that the function g defined by $g(r,\theta) = \sin\theta$ is *not* a good candidate[5] since the necessary condition of separate continuity is violated—points, including ones arbitarily close to 0, from the half-ray $\{(r,0)\colon r > 0\}$ stay at the level 0 (on W-axis), whereas points, including ones arbitrarily close to 0, from the half-ray $\{(r,\frac{\pi}{2})\colon r > 0\}$ are lifted to the level 1 on the W-axis.

Utilizing the above remarks and an easy fact that:

$$(2k+1)\frac{\pi}{2} = \frac{\pi}{2} \pmod{2\pi} \quad \text{or} \quad \frac{3\pi}{2} \pmod{2\pi}, \qquad k = 1,2,3,\dots,$$

none of the functions k, l and m given by: $k(x) = \sin 3x$, $l(x) = \sin 5x$, etc. or generally $m(x) = \sin(2k+1)x$, $k = 1,2,3,\dots$ is separately continuous and is non-continuous at $(0,0)$.

Hence, *the first* candidate to turn to is the function f given by

$$f(r,\theta) = \sin 2\theta.$$

[5] We leave as an exercise to the reader that *neither* is any rotation $g_\alpha(r,\theta) = \sin(\theta+\alpha)$ for some α, nor is $g^2(r,\theta) = \sin^2\theta$.

FIGURE 5

Observe that f meets all the above criteria, i.e., it is separately continuous; and, f is not continuous:

$$\lim_{(\frac{1}{n},\frac{\pi}{4})\to(0,0)} \sin 2\theta = 1 \neq f(0,0) = 0 .$$

But let us represent $f(r,\theta) = \sin 2\theta$ in rectangular coordinates. We will use the generally adopted xy-coordinate system. Assume $(x,y) \neq (0,0)$. Then

$$\sin 2\theta = 2\sin\theta \cdot \cos\theta = 2\frac{y}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} = \frac{2xy}{x^2+y^2},$$

so, the *modified* Heine's F function is given by:

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0. \end{cases}$$

But this is the well-known example provided by G. Peano ([GP] p. 173) in 1884, (see also [R2])!

A clear pay-off by using Peano's function f is that it has a simple form in rectangular coordinates. Let us, however, represent the original Heine's F function in rectangular coordinates. Assume $(r,\theta) \neq (0,0)$.

Then in fact:

$$\begin{aligned} -4\sin 4\theta &= -2\sin 2\theta\cdot\cos 2\theta \\ &= -2\sin 2\theta\,(1-2\sin^2\theta) \\ &= 2\sin 2\theta\,(2\sin^2\theta-1) \\ &= 2\cdot\frac{2xy}{x^2+y^2}\left(\frac{2x^2}{x^2+y^2}-1\right) \\ &= \frac{4xy}{x^2+y^2}\cdot\frac{x^2-y^2}{x^2+y^2} \\ &= \frac{4xy(x^2-y^2)}{(x^2+y^2)^2}\,. \end{aligned}$$

So, the following formula expresses Heine's F function in rectangular coordinates

$$F(x,y)=\begin{cases} \frac{4xy(x^2-y^2)}{(x^2+y^2)^2}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

Let us now return to P e a n o 's book [GP].

Although the main author of "Calcolo Differenziale" is Angelo G e n o c c h i, we can be certain that the new examples in "Calcolo" were due to Giuseppe P e a n o. In fact, here is an excerpt from the letter by Angelo G e n o c c h i (see also [Ke] p. 6):

> "... Recently the publishing firm of Bocca Brothers published a volume entitled 'Calcolo differeniziale e principii di calcolo integrale'. At the top of the title page, my name is placed ... So that nothing will be attributed to me which is not mine, I must declare that I have had no part in the compilation of the aforementioned book and that everything is due to that outstanding young man Dr. Giuseppe Peano, whose name is signed to the Preface and Annotations ...

4. Conclusion: Highlights of early discoveries in the area of separate and joint continuity

It is hard to over-emphasize the role R. B a i r e 's results [Ba] play in today's analysis and mathematics as a whole.

Additionally to the celebrated theorem mentioned in the introduction, we have:

(∗) Given a separately continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$, then f is of first class.[6]

R. Baire proved also:

(∗∗) If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is separately continuous, then there is a residual subset A of $\mathbb{R}$, such that $A \times \mathbb{R} \subset C(f)$.

This is *the first* result answering the so-called Uniformization Problem [Pi]. Nowadays theorems of this form we call *Namioka-type* theorems [Na] (see also [Pi] p. 294 and 303 for further discussion). Let us record the following two results of R. Baire:

(∗∗∗) There are separately continuous function $f\colon \mathbb{R}^3 \to \mathbb{R}$ which are discontinuous at every point of certain lines.

(∗v) Every separately continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the condition:[7]

(QC) For every point $(x_0, y_0) \in \mathbb{R}^2$, for every disc K centered at (x_0, y_0) and for every $\varepsilon > 0$, there is a disc K_1, $K_1 \subset K$, such that $|f(x,y) - f(x_0,y_0)| < \varepsilon$ for every $(x,y) \in K_1$.

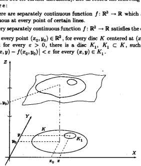

FIGURE 6

Condition (QC), mentioned above, defines the class of functions known as *quasi-continuous* (see [Ne], [Tr] or [HT]).

[6]It is the limit of a sequence $\{f_n\}$ of continuous functions $f_n\colon \mathbb{R}^2 \to \mathbb{R}$. This result has been subsequently generalized by H. Lebesgue [Le], K. Kuratowski [Ku], Montogomery [Mo], and W. Rudin [Rl], to mention only a few.

[7]As R. Baire indicates ([Ba] p. 95), condition (QC) has been suggested by Vito Volterra. I am indebted to G. Mägerl [Mä] for calling attention to this fact.

We now leave *fin de siècle* to encounter two results of H. Lebesgue [Le]. The first is a straightforward generalization of a Baire's theorem:

(v) A separately continuous function $f: \mathbb{R}^n \to \mathbb{R}$ may be of class $(n-1)$ of Baire but no worse,

and

(v*) A function $f: \mathbb{R}^2 \to \mathbb{R}$ which is separately continuous can be discontinuous at (x_0, y_0) even if f is continuous along every analytic curve through (x_0, y_0).[8]

An example of such a function can be f given by

$$f(x,y) = \begin{cases} 1, & \text{on the curve } y = e^{-\frac{1}{x^2}}, \quad \text{except for } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Along these lines, N. Lusin [Lu] proved that a function $f: [a,b] \times [c,d] \to \mathbb{R}$ is continuous if and only if its restriction to the graph of each *continuous* function $g: [a,b] \to [c,d]$ and $h: [c,d] \to [a,b]$ is continuous (see also a recent article [Da] for some strong generalizations).

We now move to two results of W. H. Young. The first one, [Yo]:

(v**) Separately continuous functions $f: \mathbb{R}^2 \to \mathbb{R}$ that are monotone in one variable are continuous,

has been re-discovered (e.g., [DK]) or is not being given credit to anybody (folklore) in real analysis textbooks.

Before we present the other result of W. M. Young, we want to draw the reader's attention to the following easy fact:

If D is a countable and dense subset of $[0,1]\times[0,1]$, then there is a separately continuous function $f: [0,1] \times [0,1] \to \mathbb{R}$ such that $D = D(f)$, where $D(f)$ stands for the set of points of discontinuity of f.

Let us sketch the arguments. Let $D = \{(x_i, y_i) : i \in \mathbb{N}\}$ and let

$$f_i(x,y) = \begin{cases} \frac{2(x-x_i)(y-y_i)}{(x-x_i)^2+(y-y_i)^2}, & \text{if } (x,y) \neq (x_i, y_i), \\ 0, & \text{if } (x,y) = (x_i, y_i). \end{cases}$$

Then, f is defined by

$$f(x,y) = \sum_{i=1}^{\infty} \frac{f_n(x,y)}{2^n}.$$

Using a family of Cantor sets in an interesting process of densifying the set $D(f)$, G. C. Young and W. H. Young [YY] show:

[8] See also [Ro].

($\ast\ast\ast$) There is a function $f\colon [0,1]\times[0,1]\to\mathbb{R}$ which is continuous with respect to every straight line and which has *uncountably* many points of discontinuity in *every* rectangle contained in the unit square.[9]

As indicated in [Pi], K. Bögel [Bö] has studied the distribution of points of continuity for $f\colon \mathbb{R}^n\to\mathbb{R}$, in hyperplanes.

Finally, last but not least — we discuss works of Hans Hahn [H1], [H2].

His monograph "Reelle Funktionen" is the first monograph, and the only so far, where the separate versus joint continuity problem receives so much attention. Fourteen pages of §39 are devoted completely to this topic.

A fact which is not generally known is that [H2] presents a metric version of the Namioka theorem!

In fact, following H. Hahn ([H2] p. 126), a space S is called an *absolute* G_δ or a *Young space*, if it is a G_δ subspace in every metric space X that contains S.[10]

Finishing, let us remark that the following result appears in [H2 §39].

($\ast\times$) Let X be a metric Young space, Y be compact metric, and let $f\colon X\times Y\to\mathbb{R}$ be separately continuous. Then there is a residual subset A of X such that $A\times Y\subset C(f)$.

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[9] In 1949 T. Tolstoff [To] showed that there is a function $f\colon\mathbb{R}^2\to\mathbb{R}$ whose set $D(f)$ of points of discontinuity has a positive Lebesgue measure.

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