

# A NOTE ON METRIC PRESERVING FUNCTIONS

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ABSTRACT. The purpose of this note is to study modifications of the Euclidean metric on  $\mathbb{R}$  with the following property: There is a monotone sequence of closed balls with empty intersection.

## 1. INTRODUCTION

**Definition 1.** We call a function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  metric preserving iff  $f(d) : M \times M \rightarrow \mathbb{R}^+$  is a metric for every metric  $d : M \times M \rightarrow \mathbb{R}^+$ , where  $(M, d)$  is an arbitrary metric space and  $\mathbb{R}^+$  denotes the set of nonnegative reals. We denote by  $\mathcal{M}$  the set of all metric preserving functions. (See Borsík [1], Borsík [2], Terpe [3].)

The following result is well known (see Borsík [1], Terpe [3]).

**Proposition 1.** *If  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a concave function vanishing exactly at origin than it is metric preserving.*

It is well known that there is a complete metric space with the following property:

There is a monotone sequence of closed balls with empty intersection. (1.1)

In Jůza [4] such a metric space (which is not discrete) has been constructed by a modification of the Euclidean metric on  $\mathbb{R}$ , where  $\mathbb{R}$  denotes the set of reals.

For each  $f \in \mathcal{M}$  denote by  $d_f$  the metric on  $\mathbb{R}$  defined as follows

$$d_f(x, y) = f(|x - y|) \text{ for each } x, y \in \mathbb{R}.$$

We call  $d_f$  a *modification* of the Euclidean metric on  $\mathbb{R}$ . (See Terpe [3].)

**Example 1.** Define  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as follows:

$$f(x) = x, \text{ if } x \leq 2, \quad f(x) = 1 + \frac{1}{x-1}, \text{ if } x > 2.$$

In Jůza [4] it is shown that  $f \in \mathcal{M}$  and the metric space  $(\mathbb{R}, d_f)$  has the property (1.1). The proof of (1.1) is based on the following property of the metric space  $(\mathbb{R}, d_f)$ :

For each compact set  $K$  there is a closed ball  $S$  and there is  
a compact set  $L$  such that  $K \subset \mathbb{R} - S \subset L$ . (1.2)

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## 2. MAIN RESULTS

**Theorem 1.** Let  $f \in \mathcal{M}$ . Suppose that there are  $g, h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $g, h$  are nonincreasing, and

$$\text{they are not constant in each neighborhood of the point } +\infty, \quad (2.1)$$

$$g(x) \leq f(x) \leq h(x) \text{ in some neighborhood of the point } +\infty, \quad (2.2)$$

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} h(x). \quad (2.3)$$

Then the metric space  $(\mathbb{R}, d_f)$  has the property (1.2).

*Proof.* Let  $m \in \mathbb{N}$  be such that  $g(x) \leq f(x) \leq h(x)$  for each  $x \in [m, \infty)$ . Put  $d = \lim_{x \rightarrow +\infty} g(x)$ . Evidently  $d = \lim_{x \rightarrow +\infty} f(x) > 0$ . Let  $K$  be a compact set. Put:  $s = \inf K - m, r = \sup K - s, \varepsilon = g(r)$ . Since  $g$  is not constant on  $(r, +\infty)$ , there is  $\xi > r$  such that  $g(\xi) \neq \varepsilon$ . Since  $g$  is nonincreasing, we have  $\varepsilon \neq g(\xi) \leq g(r) = \varepsilon$ . Therefore  $g(\xi) < \varepsilon$ . Since  $g$  is nonincreasing for each  $x \geq \xi$  we get  $g(x) \leq g(\xi)$ . Thus  $d = \lim_{x \rightarrow +\infty} g(x) \leq g(\xi) < \varepsilon$ . Let  $x \in [m, r]$ . Then  $f(x) \geq g(x) \geq g(r) = \varepsilon$ . Therefore

$$\forall x \in [m, r] : f(x) \geq \varepsilon. \quad (2.4)$$

Let  $\delta \in (d, \varepsilon)$ . Since  $\lim_{x \rightarrow +\infty} h(x) = d < \varepsilon$ , there is  $t > r$  such that  $h(t) < \delta$ . Let  $x \geq t$ . Then  $f(x) \leq h(x) \leq h(t) < \delta$ . Thus

$$\forall x \in [t, \infty) : f(x) < \delta. \quad (2.5)$$

Let  $S$  be a closed ball with the centre  $s$  and the radius  $\delta$ . Put  $L = [s - t, s + t]$ . Now, we shall show that  $K \subset \mathbb{R} - S$ . Let  $u \in K$ . Then  $|u - s| = u - s \in [m, r]$ , and by (2.4) we get  $d_f(u, s) = f(|u - s|) \geq \varepsilon > \delta$ . Therefore  $u \notin S$ . Finally, we shall show that  $\mathbb{R} - S \subset L$ . Let  $v \in \mathbb{R} - S$ . Then  $f(|v - s|) = d_f(v, s) > \delta$ . By (2.5) we have  $|v - s| < t$ . Therefore  $v \in L$ .

**Example 2.** Define  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as follows:

$$f(x) = x, \text{ if } x \in [0, 1), \quad f(x) = \frac{1 + x + \sin^2(x - 1)}{2x}, \text{ if } x \in [1, \infty).$$

It is not difficult to verify that  $f \in \mathcal{M}$  and the metric space  $(\mathbb{R}, d_f)$  has the property (1.2) (which yields also the property (1.1)), however  $f$  is not monotone on every neighborhood of the point  $+\infty$ .

**Example 3.** Define  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as follows  $f(x) = x$ , if  $x \in [0, 1]$ , and  $f(x) = \frac{1}{2} \left( x - 3n + 1 - |x - 3n + 1| + \left| x - 3n + \frac{1}{2} + \frac{1}{n+2} \right| + \left| x - 3n - \frac{1}{2} - \frac{1}{n+2} \right| \right)$ , if  $x \in (3n - 2, 3n + 1] (n = 1, 2, 3, \dots)$ . It is not difficult to verify that  $f \in \mathcal{M}$  and  $(\mathbb{R}, d_f)$  is a metric space with the property (1.1), which has not the property (1.2). Indeed, the intersection of the sequence of closed balls  $\{S_n\}_{n=1}^{\infty}$  (where  $S_n$  has the centre  $x_n = 3 \cdot (2^{n-1} - 1)$  and the radius  $\varepsilon_n = \frac{1}{2} + \frac{1}{2^{n+1}}$ ) is empty.

A characterization of metric preserving functions  $f$  such that the space  $(\mathbb{R}, d_f)$  has the property (1.1) remains an open question.

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