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THE PINCHED-CUBE TOPOLOGY

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McCoy introduced a topology intermediate between the Tikhonov and box product topologies on the countably infinite power ${}^{\omega}X$ of a topological space X. He used this topology to study Baire category in 2^X, the hyperspace of closed subsets of X in the Vietoris topology. In this note we generalize this 'pinched-cube' topology to arbitrary infinite powers, ${}^{\kappa}X$, of X and investigate the extent to which it inherits fundamental properties of X.

In §1 we introduce basic definitions and elementary facts. Separation axioms are considered in §2, compactness, connectedness, and separability in §3. In §4 we consider some completeness properties, and in §5 we explicate the connection with hyperspaces.

1. Definitions, notation, and elementary facts.

1.0. DEFINITION. Let X be a space, κ a cardinal number. By $\mathbf{S}^{\kappa}X$ we denote the set ${}^{\kappa}X$ endowed with the pinched-cube topology, which is defined as follows. For each finite set $F \subseteq \kappa$ and each function $V: F \to \tau X$ ($=$ the topology on X) let $B(F, V) = \{x \in {}^{\kappa}X: \forall\alpha \in F(x_\alpha \in V_\alpha) \wedge \forall\alpha \in \kappa(x_\alpha \in \bigcup \operatorname{ran} V)\}$. Let $\mathfrak{B}$ be the collection of all such sets $B(F, V)$. Then $\mathfrak{B}$ is a base for the desired topology. (Another way to describe $\mathfrak{B}$ is as follows. For $\varnothing \neq V \in \tau X$ let $\mathfrak{B}(V)$ be the collection of all basic open sets in the ordinary Tikhonov power ${}^{\kappa}V$; each member of $\mathfrak{B}(V)$ can be viewed as a subset of ${}^{\kappa}X$, and $\mathfrak{B} = \bigcup\{\mathfrak{B}(V): \varnothing \neq V \in \tau X\}$.)

Clearly $\mathbf{S}^{n}X$ is homeomorphic to ${}^{n}X$ with the Tikhonov topology if $n < \omega$, so we shall consider $\mathbf{S}^{\kappa}X$ only for $\kappa \geq \omega$. It is also clear that

$$\tau({}^{\kappa}X) \subseteq \tau(\mathbf{S}^{\kappa}X) \subseteq \tau(\square^{\kappa}X), \tag{1.1}$$

where $\square^{\kappa}X$ denotes ${}^{\kappa}X$ endowed with the box topology, which leads to the following observation.

1.2. *Fact*. Let $\tilde{X} = \mathbf{S}^{\kappa}X$, and for each $\alpha < \kappa$ let $A_\alpha \subseteq X$. Then

$$\operatorname{cl}_{\tilde{X}} \prod_{\alpha<\kappa} A_\alpha = \prod_{\alpha<\kappa} \operatorname{cl}_X A_\alpha. \qquad \square$$

Four subspaces of $\mathbf{S}^{\kappa}X$ will also be of interest.

1.3. DEFINITION. (a) $\Delta^{\kappa}X = \{x \in {}^{\kappa}X: x \text{ is constant}\}$.
(b) $\Phi^{\kappa}X = \{x \in {}^{\kappa}X: \exists p \in X(|\{\alpha \in \kappa: x_\alpha \neq p\}| < \omega)\}$.
(c) $\mathbf{F}^{\kappa}X = \{x \in {}^{\kappa}X: |\operatorname{ran} x| < \omega\}$.

(d) $\mathbf{C}^\kappa X = \{x \in {}^\kappa X\colon \operatorname{ran} x \text{ is a closed set in } X\}$.
Clearly $\Delta^\kappa X \subseteq \Phi^\kappa X \subseteq \mathbf{F}^\kappa X$, and $\mathbf{F}^\kappa X \subseteq \mathbf{C}^\kappa X$ if X is T_1.

1.4. *Fact*. As a subspace of $\mathbf{S}^\kappa X$, $\Delta^\kappa X \simeq X$, (where '$\simeq$' denotes homeomorphism).

1.5. *Fact*. $\Phi^\kappa X$ is a dense subspace of $\mathbf{S}^\kappa X$. If X is T_1, $\Phi^\kappa X$ is a dense subspace of $\mathbf{C}^\kappa X$.

1.6. *Fact*. For $\alpha < \kappa$ let $\pi_\alpha\colon \mathbf{S}^\kappa X \to X$ be the canonical projection map to the αth factor: $\pi_\alpha(x) = x_\alpha$. It follows from (1.1) that π_α is continuous and open, and it is easy to see that $\pi_\alpha \restriction \mathbf{F}^\kappa X$ and $\pi_\alpha \restriction \mathbf{C}^\kappa X$ are also continuous and open, while $\pi_\alpha \restriction \Delta^\kappa X$ is a homeomorphism.

2. Separation axioms.

Our first result is to be expected; it follows from (1.1).

2.0. PROPOSITION. *If X is a T_i-space, $i = 0, 1$, or 2, then so is $\mathbf{S}^\kappa X$.* □

However, $\mathbf{S}^\kappa X$ is never T_3 if X is an infinite, compact Hausdorff space, for we have the following characterization.

2.1. DEFINITION. If λ is an infinite cardinal number, we say that a space X is *λ-pseudonormal* iff whenever $H \subseteq V$, where $V \in \tau X$ and $H \in [X]^{<\lambda} \cap \sigma X$, there is a $W \in \tau X$ such that $H \subseteq W \subseteq \operatorname{cl} W \subseteq V$. (Here σX denotes the family of closed subsets of X. Clearly ω_1-pseudonormality is just the usual notion of pseudonormality, and ω-pseudonormality is equivalent to being T_3.)

2.2. THEOREM. *$\mathbf{S}^\kappa X$ is T_3 iff X is T_3 and κ^+-pseudonormal, and $[X]^{\le\kappa} \subseteq \sigma X$.*

Proof. We first prove necessity. That X is T_3 is an immediate consequence of 1.4. To prove that $[X]^{\le\kappa} \subseteq \sigma X$, let $\tilde{X} = \mathbf{S}^\kappa X$, and suppose that $C \in [X]^{\le\kappa} \setminus \sigma X$. Choose $x \in \tilde{X}$ so that $\operatorname{ran} x = C$, and pick $p \in (\operatorname{cl}_X C) \setminus C$. Let $V = X \setminus \{p\}$, and let $\tilde{V} = {}^\kappa V$. Then $x \in \tilde{V} \in \tau\tilde{X}$, and $\tilde{X}$ is T_3, so there is a basic open nbhd, $B(F, W)$, of x such that $\operatorname{cl}_{\tilde{X}} B(F, W) \subseteq \tilde{V}$. Let $G = \bigcup W[F]$; clearly $C \subseteq G$, so $p \in \operatorname{cl}_X G$. Choose $y \in \tilde{X}$ so that $y_\alpha \in W_\alpha$ if $\alpha \in F$, and $y_\alpha = p$ if $\alpha \in \kappa \setminus F$. Then $y \in (\operatorname{cl}_{\tilde{X}} B(F, W)) \setminus \tilde{V} = \varnothing$, a contradiction.

We argue similarly to prove that X is κ^+-pseudonormal. Suppose that $C \in [X]^{\le\kappa}$, $G \in \tau X$, and $C \subseteq G$. Choose $x \in \tilde{X}$ with $\operatorname{ran} x = C$, and let

$\tilde{G} = {}^{\kappa}G$. Since $\tilde{X}$ is T_3, and $x \in \tilde{G}$, x has a basic open nbhd, $B(F, V)$, whose closure is contained in $\tilde{G}$. But then if $W = \bigcup V[F]$, clearly $C \subseteq W \subseteq \mathrm{cl}_X W \subseteq G$, as required.

We now show sufficiency. Fix $x \in \tilde{X}$, and let $B(F, V)$ be a basic open nbhd of x. Let $C = \operatorname{ran} x$, let $W = \bigcup V[F]$. C is a closed, discrete subset of X, $|C| \leq \kappa$, and $C \subseteq W$. Thus, there is an open $G \subseteq X$ such that $C \subseteq G \subseteq \mathrm{cl}_X G \subseteq W$. Choose $\beta \in \kappa \setminus F$ arbitrarily, let $F' = F \cup \{\beta\}$, and choose $U\colon F' \to \tau X$ so that for each $\alpha \in F$, $x_\alpha \in U_\alpha \subseteq G \cap \mathrm{cl}_X U_\alpha \subseteq V_\alpha$, and $U_\beta = G$. Then $x \in B(F', U) \subseteq \mathrm{cl}_{\tilde{X}} B(F', U) \subseteq B(F, V)$, so $\tilde{X}$ is T_3. □

(The condition that X be κ^+-pseudonormal with $[X]^{\leq\kappa} \subseteq \sigma X$ is a kind of strengthened T_3 separation axiom: X is ω-pseudonormal iff X is regular, and $[X]^{<\omega} \subseteq \sigma X$ iff X is T_1.)

Observe that if $\mathbf{S}^\kappa X$ is T_3, then in fact $\mathbf{S}^\kappa X = \mathbf{C}^\kappa X$. In general, $\mathbf{C}^\kappa X$ behaves better than $\mathbf{S}^\kappa X$ with respect to the higher separation axioms, as the following result attests.

2.3. THEOREM. *If X is T_4, then $\mathbf{C}^\kappa X$ is Tikhonov.*

Proof. Let $\tilde{X} = \mathbf{C}^\kappa X$. Fix $x \in \tilde{X}$, and let $B(F, V)$ be a basic open nbhd of x (in $\tilde{X}$: this ambiguity in the notation '$B(F, V)$' should cause no confusion). Let $C = \operatorname{ran} x$, and let $W = \bigcup V[F]$; C is closed, and $C \subseteq W \in \tau X$, so by normality of X there is a function $f \in C(X, [0, 1])$ such that $f[C] = \{0\}$ and $f[X \setminus W] = \{1\}$. Let $F = \{\alpha_0, \ldots, \alpha_{n-1}\}$, and for $i < n$ let $f_i \in C(X, [0, 1])$ be such that $f_i(x_{\alpha_i}) = 0$ and $f_i[X \setminus V_{\alpha_i}] = \{1\}$. Let $h = f \wedge \bigwedge\{f_i\colon i < n\}$, the minimum of the functions chosen. Then $h \in C(X, [0, 1])$, $h[C] = \{0\}$, $h[X \setminus W] = \{1\}$, and $h \leq f_i$ for $i < n$. Finally define

$$\tilde{h}\colon \tilde{X} \to [0, 1] : y \mapsto \sup\bigl(\{f_i(y_{\alpha_i})\colon i < n\} \cup \{h(y_\alpha)\colon \alpha \in \kappa \setminus \{F\}\}\bigr).$$

Fix $r \in [0, 1]$ and $y \in \tilde{X}$. Then $\tilde{h}(y) > r$ iff either $h(y_\alpha) > r$ for some $\alpha \in \kappa \setminus F$, or $f_i(y_{\alpha_i}) > r$ for some $i < n$. Clearly, then, $\tilde{h}^{-1}[(r, 1]]$ is open in $\tilde{X}$; in fact, it is even open in the Tikhonov topology on $\tilde{X}$ as a subspace of ${}^{\kappa}X$. Thus, $\tilde{h}$ is lower semi-continuous.

Now $\tilde{h}(y) < r$ iff $g(y) = \max\{f_i(y_{\alpha_i})\colon i < n\} < r$ *and* $k(y) = \sup\{h(y_\alpha)\colon \alpha \in \kappa \setminus F\} < r$. For each $m \in \omega$ let $G_m = \{y \in \tilde{X}\colon g(y), k(y) < r - 2^{-m}\}$; clearly $\tilde{h}^{-1}[[0, r)] = \bigcup\{G_m\colon m \in \omega\}$. For each $i < n$ and $m \in \omega$ let $H^m_{\alpha_i} = f_i^{-1}[[0, r - 2^{-m})]$. Fix $\alpha_n \in \kappa \setminus F$, let $F' = F \cup \{\alpha_n\}$, and let $H^m_{\alpha_n} = h^{-1}[[0, r - 2^{-m})]$ for each $m \in \omega$. Since each $f_i \geq h$, $H^m_{\alpha_i} \subseteq H^m_{\alpha_n}$ for $i < n$; thus, $B(F', H^m)$ is a basic open set for each $m \in \omega$. And if $z \in B(F', H^m)$, then clearly $g(z) < r - 2^{-m}$ and $k(z) \leq r - 2^{-m}$, so

$z \in G_{m+1}$, i.e., $B(F', H^m) \subsetneq G_{m+1}$. On the other hand, $G_m \subseteq B(F', H^{m+1})$. Thus, $\tilde{h}^{-1}[[0, r)] = \bigcup\{G_m : m \in \omega\} = \bigcup\{B(F', H^m) : m \in \omega\}$ is open, and $\tilde{h}$ is continuous.

Obviously $\tilde{h}(x) = 0$. Suppose that $y \in \tilde{X} \setminus B(F, V)$. If $y_\alpha \in X \setminus W$ for some $\alpha \in \kappa \setminus F$, then $h(y_\alpha) = 1$, so $k(y) = 1$, and $\tilde{h}(y) = 1$ also; otherwise, $y_{\alpha_i} \in X \setminus V_{\alpha_i}$ for some $i < n$, but then $f_i(y_{\alpha_i}) = 1$, $g(y) = 1$, and $\tilde{h}(y) = 1$ again. That is, $\tilde{h}$ separates x from $\tilde{X} \setminus B(F, V)$, so $\tilde{X}$ is Tikhonov. □

2.4. Example. *A metric space, X, such that $\mathbf{S}^\omega X = \mathbf{C}^\omega X$, but $\mathbf{S}^\omega X$ is not normal.* X is just a countably infinite discrete space. However, for convenience we take $X = {}^{<\omega}\omega$ with the discrete topology, where ${}^{<\omega}\omega = \bigcup\{{}^n\omega : n \in \omega\}$. Let $\tilde{X} = \mathbf{S}^\omega X$. For each $x \in {}^\omega\omega$ let $\tilde{x} \in \tilde{X}$ be defined by: $\tilde{x}_n = x \restriction n$. Let $D = \{\tilde{x} \in \tilde{X} : x \in {}^\omega\omega\}$. Clearly $|D| = 2^\omega$.

If $y \in \tilde{X} \setminus D$, then either $y_n \notin {}^n\omega$ for some $n \in \omega$, or $y_m \restriction n \neq y_n$ for some $m, n \in \omega$ with $n < m$, and in either case y has an open nbhd disjoint from D. (In fact, y has such a nbhd open in the Tikhonov topology on ${}^\omega X$.) Thus, D is closed.

Now fix $x \in {}^\omega\omega$, let $V = \operatorname{ran} \tilde{x}$, and let $\tilde{V} = {}^\omega V$; clearly $\tilde{x} \in \tilde{V} \in \tau\tilde{X}$. But if $y \in {}^\omega\omega \setminus \{x\}$, then $y_n \neq x_n$ for some $n \in \omega$, and therefore $\tilde{y}_m \notin V$ for all $m > n$, i.e., $\tilde{y} \notin \tilde{V}$. Hence D is also discrete.

Finally, $\Phi^\omega X$ is a countable, dense subset of $\tilde{X}$, so $\tilde{X}$ is separable, and it follows immediately from Jones's Lemma that $\tilde{X}$ is not normal. □

Observe that the full strength of normality of X is not used in the proof of 2.3 unless $\kappa = |X|$: we use only the fact that if $C \in [X]^{\leq\kappa} \cap \sigma X$, and $C \subsetneq V \in \tau X$, then there is an $f \in C(X)$ such that $f[C] = \{0\}$ and $f[X \setminus V] = \{1\}$. This observation paves the way for the following example.

2.5. Example. *A non-normal space, X, such that $\mathbf{C}^\omega X$ is Tikhonov.* Let $X = \omega_1 \times (\omega_1 + 1)$, where ω_1 and $\omega_1 + 1$ are given their respective order topologies. It is well known that X is not normal: e.g., $\omega_1 \times \{\omega_1\}$ and $\{\langle\alpha, \alpha\rangle : \alpha \in \omega_1\}$ are disjoint closed sets in X which cannot be separated by disjoint open sets. However, it is easy to verify that if H and K are disjoint closed subsets of X, and $|H| \leq \omega$, then there is a clopen set, C, such that $H \subsetneq C \subsetneq X \setminus K$. [Find a clopen subset, C_0, of ω_1 such that $H \cap T \subsetneq C_0 \times \{\omega_1\} \subsetneq X \setminus K$, where $T = \omega_1 \times \{\omega_1\}$. For each $\alpha \in C_0$ there is a $\beta_\alpha \in \omega_1$ such that if $\gamma < \omega_1$ and $\langle\alpha, \gamma\rangle \in H \cup K$, then $\gamma < \beta_\alpha$. Let $\beta = \sup\{\beta_\alpha : \alpha \in C_0\}$, and let $C_1 = C_0 \times (\beta, \omega_1]$; C_1 is clopen, and $H \cap T \subsetneq C_1 \subsetneq X \setminus K$. Moreover, there is a $\gamma < \omega_1$ such that $H \setminus T \subsetneq (\gamma + 1) \times (\beta + 1)$, a clopen set disjoint from C_1. $(\gamma + 1) \times (\beta + 1)$ is a countable, compact metric space, so it contains a clopen set C_2 such that $H \setminus T \subsetneq C_2 \subsetneq X \setminus K$. Now let $C = C_1 \cup C_2$.] The function f defined by $f[C] = \{0\}$ and $f[X \setminus C] = \{1\}$ is continuous, and it follows from the observation immediately preceding this example that $\mathbf{C}^{\omega}X$ is Tikhonov. □

Let us call a space X *functionally κ-pseudonormal* iff whenever $H \in [X]^{<\kappa} \cap \sigma X$, $K \in \sigma X$, and $H \cap K = \varnothing$, there is an $f \in C(X)$ such that $f[H] = \{0\}$ and $f[K] = \{1\}$.

2.6. *Question*. If $\mathbf{C}^{\kappa}X$ is Tikhonov, must X be functionally κ^{+}-pseudonormal?

3. Compactness, connectedness, and separatility. If X is a non-degenerate Hausdorff space, then $\tau(\mathbf{S}^{\kappa}X) \supset \tau({}^{\kappa}X)$, so $\mathbf{S}^{\kappa}X$ cannot be compact: its topology is too fine. However, we can say much more.

3.0. THEOREM. $\mathbf{S}^{\kappa}X$ *is (countably) compact* iff (i) *X is (countably) compact, and* (ii) *X does not contain two disjoint, non-empty closed sets (i.e., $\sigma X \setminus \{\varnothing\}$ is a filter-base).*

Proof. We first prove necessity. If X were not (countably) compact, there would be a (countable) filter-base $\mathfrak{F} \subseteq \sigma X \setminus \{\varnothing\}$ with $\bigcap \mathfrak{F} = \varnothing$. But then $\{{}^{\kappa}F\colon F \in \mathfrak{F}\}$ would have the same properties in $\tilde{X} = S^{\kappa}X$, contradicting the (countable) compactness of $\tilde{X}$. This proves (i). To prove (ii), suppose that $F_0, F_1 \in \sigma X \setminus \{\varnothing\}$ are disjoint, and let $Y = \{x \in \tilde{X}\colon \forall \alpha \in \kappa \setminus \omega (x_\alpha \in F_0)\}$. Let $V = X \setminus F_0$, and for $n \in \omega$ let $W(n) = \{x \in Y\colon x_n \in V\}$. Let $W(\omega) = [{}^{\kappa}(X \setminus F_1)] \cap Y$. Then $\{W(\alpha)\colon \alpha \leq \omega\}$ is an open cover of Y with no finite subcover, and Y is closed in $\tilde{X}$, so $\tilde{X}$ is not countably compact.

To prove sufficiency, observe that (i) and (ii) imply that $\sigma X \setminus \{\varnothing\}$ is a countably complete filter-base on X. But then $\{\mathrm{cl}_{\tilde{X}}\{x\}\colon x \in \tilde{X}\}$ is easily seen to be a base for a countably complete filter on $\tilde{X}$. In particular, for each $C \in [\tilde{X}]^{\omega}$, $\bigcap\{\mathrm{cl}_{\tilde{X}}\{x\}\colon x \in C\} \neq \varnothing$, so C has an accumulation point in $\tilde{X}$, and $\tilde{X}$ is therefore countably compact. And if X is compact, then clearly $\bigcap\{\mathrm{cl}_X\{x\}\colon x \in X\} \neq \varnothing$, so $\bigcap\{\mathrm{cl}_{\tilde{X}}\{x\}\colon x \in \tilde{X}\} \neq \varnothing$, and $\tilde{X}$ is therefore compact. □

Essentially the same argument can be carried out with $\mathbf{S}^{\kappa}X$ replaced by $\mathbf{\Phi}^{\kappa}X$ or $\mathbf{F}^{\kappa}X$, though not with $\mathbf{C}^{\kappa}X$, since the sets $W(\alpha) (\alpha \leq \omega)$ may constitute a *finite* cover of $Y \cap \mathbf{C}^{\kappa}X$: some of them may be empty. (In fact, $\mathbf{C}^{\kappa}X$ may be empty: take $X = \omega_1 \times 2$, $\tau X = \{\varnothing, \omega_1 \times \{0\}, \omega_1 \times \{1\}, X\}$, and $\kappa = \omega$.) Thus we have the following corollary to the proof of 3.0.

3.1. COROLLARY. *The following are equivalent*:
(a) $\mathbf{S}^{\kappa}X$ *is countably compact*;
(b) $\mathbf{F}^{\kappa}X$ *is countably compact*;
(c) $\Phi^{\kappa}X$ *is countably compact*;
(d) $\sigma X \setminus \{\emptyset\}$ *is a countably complete filter-base on* X. $\square$

We assume henceforth that X *is* T_1. In particular, $\mathbf{S}^{\kappa}X$, $\mathbf{C}^{\kappa}X$, $\mathbf{F}^{\kappa}X$, and $\Phi^{\kappa}X$ are never countably compact unless X is degenerate.

We have not investigated the circumstances under which $\mathbf{S}^{\kappa}X$, say, is Lindelöf, except to note the following non-trivial example. Take X to be the discrete two-point space. Then $\mathbf{S}^{\omega}X$ is Lindelöf, because it is homeomorphic to the space obtained from the (middle-thirds) Cantor Space by isolating the points 0 and 1.

We now consider connectedness.

3.2. THEOREM. *The following are equivalent*:
(a) X *is connected*;
(b) $\Phi^{\kappa}X$ *is connected*;
(c) $\mathbf{F}^{\kappa}X$ *is connected*;
(d) $\mathbf{C}^{\kappa}X$ *is connected*;
(e) $\mathbf{S}^{\kappa}X$ *is connected*.

Proof. (e) → (a). If H is a non-empty, proper clopen subset of X, then ${}^{\kappa}H$ is a non-empty, proper clopen subset of $\mathbf{S}^{\kappa}X$.

(a) → (b). For each $F \in [\kappa]^{<\omega}$ let $\Delta_F = \{x \in \Phi^{\kappa}X;\ \exists p \in X \forall \alpha \in \kappa \setminus F (x_\alpha = p)\}$; clearly $\Phi^{\kappa}X = \bigcup \{\Delta_F\colon F \in [\kappa]^{<\omega}\}$. Moreover, Δ_F is homeomorphic to ${}^{|F|}X \times X$ endowed with a topology slightly coarser than the usual product topology, so — since ${}^{|F|}X$ and X are connected — so is Δ_F. Finally, $\bigcap \{\Delta_F\colon F \in [\kappa]^{<\omega}\} = \Delta^{\kappa}X (= \Delta_{\emptyset})$ is connected, so $\Phi^{\kappa}X$ is connected.

(b) → (c) → (d) → (e). Each space is dense in the next. (Recall that X is T_1.) $\square$

It is well known that $\square^{\omega}R$ is not connected. Thus, in respect of connectedness $\mathbf{S}^{\kappa}X$ resembles ${}^{\kappa}X$ more than it does $\square^{\kappa}X$. The situation is similar when we consider separability (or, more generally, density): $\square^{\omega}\omega$ is a discrete space of power 2^{ω}, but $\mathbf{S}^{\omega}\omega$, like ${}^{\omega}\omega$, is separable. In fact, we have the following result.

3.3. THEOREM. *Let* $\lambda = d(X)$, *the density of* X. (I.e., $d(X) = \omega \cdot \min\{|D|\colon D$ *is a dense subset of* $X\}$.) *For any* $\kappa \leq 2^{\lambda}$, $d(\mathbf{S}^{\kappa}X) = d(\mathbf{C}^{\kappa}X) = d(\mathbf{F}^{\kappa}X) = d(\mathbf{F}^{\kappa}X) = d(X) = d({}^{\kappa}X)$.

Proof. It is well known (e.g., [3, 4.5]) that $d(X) = d({}^{\kappa}X)$ if $\kappa \le 2^{\lambda}$. Close examination of the proof given in [Ju] shows, however, that $d(X) = d(\mathbf{F}^{\kappa}X)$ as well: only obvious minor modifications are required. The remaining two equalities are obvious. □

We may also consider the cellularity of $\mathbf{S}^{\kappa}X$. (Recall that $c(X)$, the *cellularity* of X, is $\omega \cdot \sup\{|\mathcal{V}| : \mathcal{V} \subseteq \tau X \setminus \{\varnothing\}$ and $\mathcal{V}$ is disjoint$\}$.) We shall restrict our attention to ccc spaces, however, i.e., spaces X for which $c(X) = \omega$.

Clearly $\square^{\omega}X$ is never ccc if X has two disjoint, non-empty open sets. On the other hand, if ${}^{n}X$ is ccc for each $n \in \omega$, then ${}^{\kappa}X$ is ccc for all κ [7, II(8)]. In particular, ${}^{\kappa}X$ is always ccc if X is separable. It is also known [7, III(7)] that Martin's Axiom together with the negation of the Continuum Hypothesis ($MA + \neg CH$) implies that every (finite) product of ccc spaces is ccc. On the other hand, it is consistent with the usual axioms of set theory that there be a Suslin Line [7, VI(4)], which is *inter alia* a ccc space whose square is not ccc. Thus, we cannot hope to show outright that, say, $\mathbf{S}^{\omega}X$ is ccc if X is. We consider, instead, a slightly stronger property.

3.4. Definition. A space X is said to have *Property* K iff every uncountable family of non-empty open subsets of X has an uncountable linked subfamily. (A collection of sets is *linked* iff no two of its members are disjoint.) A space with Property K is obviously ccc.

Property K is (in the usual sense) productive, but $\square^{\kappa}X$ has Property K iff $\tau X \setminus \{\varnothing\}$ is a filter-base. Here again the pinched-cube topology follows the Tikhonov topology rather than the box topology.

3.5. Theorem. *If X has Property* K, *so do* $\mathbf{S}^{\kappa}X$, $\mathbf{C}^{\kappa}X$, $\mathbf{F}^{\kappa}X$, *and* $\Phi^{\kappa}X$.

Proof. It suffices to prove that $\Phi^{\kappa}X$ has Property K, since $\Phi^{\kappa}X$ is dense in each of the other spaces. Let $\tilde{X} = \Phi^{\kappa}X$, and let $\mathcal{U} = \{U_{\alpha}: \alpha < \omega_1\} \subseteq \tau\tilde{X} \setminus \{\varnothing\}$; we must extract from $\mathcal{U}$ an uncountable, linked sub-family.

There is no harm in assuming that each U_{α} is a basic open set. However, it will be convenient to modify slightly our notation for such sets: we write $U_{\alpha} = \tilde{B}(V_{\alpha}, \sigma_{\alpha}, W_{\alpha})$, where

(i) $V_{\alpha} \in \tau X$;
(ii) $\operatorname{dom} \sigma_{\alpha} = \operatorname{dom} W_{\alpha} = n_{\alpha}$ for some $n_{\alpha} \in \omega$;
(iii) $\sigma_{\alpha}: n_{\alpha} \to \kappa$ and is 1-1;
(iv) $W_{\alpha}: n_{\alpha} \to \tau X \setminus \{\varnothing\}$;

(v) $W_\alpha(i) \subseteq V_\alpha$ for each $i < n_\alpha$; and

$$\tilde{B}(V_\alpha, \sigma_\alpha, W_\alpha) = \left\{x \in \tilde{X} \cap {}^\kappa V_\alpha : \forall i < n_\alpha\left(x_{\sigma_\alpha(i)} \in W_\alpha(i)\right)\right\}.$$

By the Δ-System Lemma [3, A2.2] there are sets $F \in [\kappa]^{<\omega}$ and $I_0 \in [\omega_1]^{\omega_1}$ such that $\operatorname{ran} \sigma_\alpha \cap \operatorname{ran} \sigma_\beta = F$ whenever $\alpha, \beta \in I_0$ with $\alpha \neq \beta$. Clearly there are $n \in \omega$ and $I_1 \in [I_0]^{\omega_1}$ such that $n_\alpha = n$ for each $\alpha \in I_1$. For each $\alpha \in I_1$ let $F_\alpha = (\operatorname{ran} \sigma_\alpha) \setminus F$. By composing σ_α and W_α ($\alpha \in I_1$) with a permutation of n if necessary — this does not change U_α — we may assume that for some $m \leq n$ and $\phi: n \setminus m \to F$ we have $\sigma_\alpha[m] = F_\alpha$ and $\sigma_\alpha \restriction n \setminus m = \phi$ for each $\alpha \in I_1$. Finally, we apply Property K to find an $I_2 \in [I_1]^{\omega_1}$ such that $\{V_\alpha : \alpha \in I_2\}$ and each $\{W_\alpha(i) : \alpha \in I_2\}$ ($i < n$) are linked families.

Now suppose that $\alpha, \beta \in I_2$ with $\alpha \neq \beta$. For each $i < n$ fix $p_i \in W_\alpha(i) \cap W_\beta(i)$, and choose also a point $p \in V_\alpha \cap V_\beta$. Define $x \in \tilde{X}$ by

$$x_\xi = \begin{cases} p, & \text{if } \xi \in \kappa \setminus (F \cup F_\alpha \cup F_\beta), \\ p_i, & \text{if } \xi = \sigma_\alpha(i) \text{ or } \xi = \sigma_\beta(i). \end{cases}$$

To see that x is well-defined, note that if $\xi \in F$, then there is a unique $i \in n \setminus m$ such that $\xi = \sigma_\alpha(i) = \sigma_\beta(i) = \phi(i)$, and that F, F_α, and F_β are disjoint. Clearly $x \in U_\alpha \cap U_\beta$, so $\{U_\alpha : \alpha \in I_2\}$ is linked. □

3.6. COROLLARY. [*MA* + ¬*CH*] *If* X *is ccc, so are* $\Phi^\kappa X$, $F^\kappa X$, $C^\kappa X$, *and* $S^\kappa X$.

Proof. $MA + \neg CH$ implies that ccc and Property K are equivalent [3, 5.3]. □

4. Completeness properties.

The fundamental completeness property is probably that of being a Baire space. Unfortunately, there are Baire spaces whose squares are not Baire [2], so we shall consider instead some stronger properties which are productive. We begin by recalling some definitions.

4.0. DEFINITION (1). Let X be a topological space. A function Φ: $\tau X \setminus \{\emptyset\} \to \tau X \setminus \{\emptyset\}$ is called a *winning strategy* iff (1) $\Phi(V) \subseteq V$ for each $V \in \tau X \setminus \{\emptyset\}$, and (2) $\bigcap\{V_n : n \in \omega\} \neq \emptyset$ whenever $\langle V_n : n \in \omega\rangle$ is a sequence in $\tau X \setminus \{\emptyset\}$ satisfying $V_{n+1} \subseteq \Phi(V_n)$ for each $n \in \omega$. X is said to be *α-favorable* iff X has a winning strategy.

4.1. DEFINITION (1). Let X be a topological space, $\mathcal{P} = \{\langle x, V\rangle \in X \times \tau X : x \in V\}$. A function Φ: $\mathcal{P} \to \tau X \setminus \{\emptyset\}$ is called a *strong winning strategy* iff (1) $x \in \Phi(\langle x, V\rangle) \subseteq V$ for each $\langle x, V\rangle \in \mathcal{P}$, and (2)

$\bigcap\{V_n: n \in \omega\} \neq \varnothing$ whenever $\langle\langle x_n, V_n\rangle: n \in \omega\rangle$ is a sequence in $\mathcal{P}$ satisfying $V_{n+1} \subseteq \Phi(x_n, V_n)$ for each $n \in \omega$. X is said to be *strongly α-favorable* iff X has a strong winning strategy.

4.2. DEFINITION (8). Let X be a topological space. A *weak strategy* on X is a sequence, $\langle \Phi_n: n \geq 1\rangle$, of functions such that

(a) Φ_n: ${}^n(\tau X \setminus \{\varnothing\}) \to \tau X \setminus \{\varnothing\}$ for each $n \geq 1$, and

(b) whenever $\langle V_0, \ldots, V_{n-1}\rangle \in \operatorname{dom} \Phi_n$, then $\Phi_n(\langle V_0, \ldots, V_{n-1}\rangle) \subseteq V_{n-1}$.

A sequence $\langle V_n: n < \alpha\rangle$ (where $0 < \alpha \leq \omega$) is compatible (with the weak strategy) iff $V_k \subseteq \Phi_k(\langle V_0, \ldots, V_{k-1}\rangle)$ whenever $1 \leq k < \alpha$. The weak strategy is a *weak winning strategy* iff $\bigcap\{V_n: n \in \omega\} \neq \varnothing$ whenever $\langle V_n: n \in \omega\rangle$ is compatible. X is said to be weakly *α-favorable* iff X has a weak winning strategy.

4.3. DEFINITION. X is *π-regular* (called quasi-regular in [6]) iff the regularly closed subsets of X form a π-base for C, i.e., iff for each $V \in \tau X \setminus \{\varnothing\}$ there is a $W \in \tau X \setminus \{\varnothing\}$ with $\operatorname{cl} W \subseteq V$. X is *pseudo-complete* (6) iff (1) X is π-regular, and (2) X has a sequence, $\langle \mathfrak{B}_n: n \in \omega\rangle$, of open π-bases such that $\bigcap\{V_n: n \in \omega\} \neq \varnothing$ whenever $\operatorname{cl} V_{n+1} \subseteq V_n \in \mathfrak{B}_n$ for each $n \in \omega$.

Combining results of [6], [1], and [8], we see that:

$$\begin{array}{ccccccc} \text{strongly } \alpha\text{-favorable} & \rightarrow & \alpha\text{-favorable} & \rightarrow & \text{weakly } \alpha\text{-favorable} & \rightarrow & \text{Baire} \\ & & & \nearrow & & & \\ & & \text{pseudo-complete} & & & & \end{array}$$

Moreover, each of these four completeness properties is preserved by arbitrary Tikhonov or box products, and the α-favorability properties are preserved by open, continuous surjections.

We find it useful in this section to extend our notation concerning basic open sets in $\mathbf{S}^\kappa X$. If B is such a set, and $\alpha \in \kappa$, we let $B_\alpha = \pi_\alpha[B]$; we denote by $U(B)$ the 'uniform part' of B, i.e., $\bigcup\{B_\alpha: \alpha \in \kappa\}$; and we let $\operatorname{supp}(B) = \{\alpha \in \kappa: B_\alpha \neq U(B)\}$, the *support* of B. $\tilde{\mathfrak{B}}$ denotes the family of non-empty basic open sets in $\mathbf{S}^\kappa X$.

4.4. THEOREM. *Let X be a strongly α-favorable T_2-space; then so is $\mathbf{S}^\kappa X$.*

Proof. Let $\tilde{X} = \mathbf{S}^\kappa X$, $\mathcal{P} = \{\langle x, V\rangle \in X \times \tau X: v \in V\}$, and $\tilde{\mathcal{P}} = \{\langle x, V\rangle \in \tilde{X} \times \tau\tilde{X}: x \in V\}$. Let Φ: $\mathcal{P} \to \tau X \setminus \{0\}$ be a strong winning strategy on X. For each $B \in \tilde{\mathfrak{B}}$ let $I(B) = \{\alpha \in \operatorname{supp}(B): B_\alpha \cap B_\beta = \varnothing$

for some $\beta \in \operatorname{supp}(B)\}$, and note that if $B \supseteq W \in \mathfrak{B}$, then $I(B) \subseteq \operatorname{supp}(W)$. We now construct a strong winning strategy, $\tilde{\Phi}$, in $\tilde{X}$.

First, for each $\langle x, V\rangle \in \tilde{\mathfrak{P}}$ choose $B(x, V) \in \tilde{\mathfrak{B}}$ so that $x \in B(x, V) \subseteq V$. Now fix $\langle x, V\rangle \in \tilde{\mathfrak{P}}$, and let $B = B(x, V)$. We shall define a set $\tilde{B} \in \tilde{\mathfrak{B}}$ by defining $\tilde{B}_\alpha$ for each $\alpha \in \kappa$; $\tilde{\Phi}(\langle x, V\rangle)$ will then be $\tilde{B}$.

If $x \in \Delta^\kappa X$, let $W = \bigcap\{B_\alpha\colon \alpha \in \kappa\}$; clearly $\langle x, W\rangle \in \mathfrak{P}$, so we may put $\tilde{B}_\alpha = \Phi(\langle x, W\rangle)$ for each $\alpha \in \kappa$.

If $x \in \tilde{X} \setminus \Delta^\kappa X$ there is a finite set $J \subseteq \kappa$ such that $\operatorname{supp}(B) \subseteq J$ and $x \restriction J$ is not constant. Choose sets G_α ($\alpha \in J$) such that

(a) $\langle x_\alpha, G_\alpha\rangle \in \mathfrak{P}$;
(b) $G_\alpha \subseteq B_\alpha$; and
(c) $G_\alpha \cap G_\beta = \varnothing$ for some $\beta \in J$

for each $\alpha \in J$. (Getting (c) is possible because X is T_2.) Then let

$$\tilde{B}_\alpha = \begin{cases} \Phi(\langle x_\alpha, G_\alpha\rangle), & \alpha \in J \\ B_\alpha, & \alpha \in \kappa \setminus J. \end{cases}$$

Clearly $x \in \tilde{B} \in \tilde{\mathfrak{B}}$. Moreover, if $\tilde{B} \supseteq W \in \tilde{\mathfrak{B}}$, then $\operatorname{supp}(W) \supseteq I(\tilde{B}) = \operatorname{supp}(\tilde{B}) \supseteq J \supseteq \operatorname{supp}(B)$.

Now fix a sequence $\langle\langle x^n, V^n\rangle\colon n \in \omega\rangle$ in $\tilde{\mathfrak{P}}$ such that for each $n \in \omega$, $V^{n+1} \subseteq \tilde{\Phi}(\langle x^n, V^n\rangle)$. For each $n \in \omega$ let $B^n = B(x^n, V^n)$, and let $\tilde{B}^n = \tilde{\Phi}(\langle x^n, V^n\rangle)$. Let $C = \bigcap\{V^n\colon n \in \omega\} = \bigcap\{\tilde{B}^n\colon n \in \omega\}$, and let $C_\alpha = \pi_\alpha[C]$ for $\alpha \in \kappa$. (Clearly $C = \Pi\{C_\alpha\colon \alpha \in \kappa\}$.) We must show that $C \neq \varnothing$.

Let $N = \{n \in \omega\colon x^n \in \Delta^\kappa X\}$. If $|N| = \omega$, then $C = \bigcap\{\tilde{B}^n\colon n \in N\}$, so $C_\alpha = \bigcap\{\tilde{B}^n_\alpha\colon n \in N\}$ for each $\alpha \in \kappa$. But for any $n \in N$ and $\alpha \in \kappa$, $\tilde{B}^n_\alpha = \Phi(\langle x^n_0, U(B^n) \cap \bigcap\{B^n_\beta\colon \beta \in \kappa\}\rangle)$, so it follows from the choice of Φ that $C_\alpha \neq \varnothing$, and hence that $C \neq \varnothing$.

Otherwise $|N| < \omega$, and we may assume that $N = \varnothing$. Let $K = \bigcup\{\operatorname{supp}(\tilde{B}^n)\colon n \in \omega\} = \bigcup\{I(\tilde{B}^n)\colon n \in \omega\} = \bigcup\{\operatorname{supp}(B^n)\colon n \in \omega\}$. Fix $\alpha \in K$. There is a $k \in \omega$ such that $\alpha \in \operatorname{supp}(\tilde{B}^n)$ for all $n \geq k$. Thus, if $n \geq k$, then $\tilde{B}^n_\alpha = \Phi(\langle x^n_\alpha, G^n_\alpha\rangle)$ for some $G^n_\alpha \subseteq B^n_\alpha$ with $x^n_\alpha \in G^n_\alpha$, and it follows as before that $C_\alpha = \bigcap\{\tilde{B}^n_\alpha\colon n \geq k\} \neq \varnothing$. And if $\alpha \in \kappa \setminus K$, then obviously $C_\alpha \supseteq C_\beta$ for any $\beta \in K$ (since $\tilde{B}^n_\alpha \supseteq \tilde{B}^n_\beta$ for each $n \in \omega$), so $C_\alpha \neq \varnothing$ in this case as well. Thus, $C \neq \varnothing$, and $\tilde{X}$ is strongly α-favorable. □

4.5. *Question*. Is the restriction to Hausdorff spaces in 4.4 necessary?

Essentially the same proof yields a better result for α-favorability.

4.6. THEOREM. *If X is α-favorable, so is* $S^\kappa X$.

Proof. Let $\tilde{X} = S^\kappa X$, and let Φ be a winning strategy on X. Suppose that $B \in \mathfrak{B}$. If $\bigcap \{B_\alpha : \alpha \in \kappa\} \neq \varnothing$, let $B^* = {}^\kappa(\bigcap \{B_\alpha : \alpha \in \kappa\})$. Otherwise we can find a $k \geq 2$ and a partition, $\{I_0, \ldots, I_{k-1}\}$, of supp(B) such that if we set $G_i = \bigcap \{B_\alpha : \alpha \in I_i\}$ for each $i < k$, then each $G_i \neq \varnothing$, and $G_i \cap G_j = \varnothing$ whenever $i < j < k$. We then define $B^* \in \mathfrak{B}$ by letting

$$B^*_\alpha = \begin{cases} G_i, & \text{if } \alpha \in I_i \text{ for some } i < k \\ \bigcup_{i<k} G_i, & \text{if } \alpha \in \kappa \setminus \operatorname{supp}(B). \end{cases}$$

Clearly $B \supseteq B^* \in \mathfrak{B}$ for each $B \in \mathfrak{B}$. For each $V \in \tau\tilde{X} \setminus \{\varnothing\}$ fix a $B(V) \in \mathfrak{B}$ such that $B(V) = (B(V))^* \subseteq V$.

Now fix $V \in \tau\tilde{X} \setminus \{\varnothing\}$, and let $B = B(V)$. As before we define $\tilde{B} = \tilde{\Phi}(V)$ by defining $\tilde{B}_\alpha$ for each $\alpha \in \kappa$. If there is a $G \in \tau X \setminus \{\varnothing\}$ such that $B_\alpha = G$ for each $\alpha \in \kappa$, we let $\tilde{B}_\alpha = \Phi(G)$ for each $\alpha \in \kappa$. Otherwise, there is a partition, $\{I_0, \ldots, I_{k-1}\}$, of supp(B), and there are disjoint sets $G_i \in \tau X \setminus \{\varnothing\}$ for $i < k$, such that $k \geq 2$, $B_\alpha = G_i$ if $\alpha \in I_i$ for some $i < k$, and $B_\alpha = \bigcup\{G_i : i < k\}$ otherwise. In this case we simply let

$$\tilde{B}_\alpha = \begin{cases} \Phi(G_i), & \text{if } \alpha \in I_i \text{ for some } i < k \\ \bigcup_{i<k} \Phi(G_i), & \text{otherwise,} \end{cases}$$

noting that supp$(\tilde{B})$ = supp(B) in both cases.

Now suppose that $\langle V^n : n \in \omega \rangle$ is a sequence in $\tau\tilde{X} \setminus \{\varnothing\}$ such that $V^{n+1} \subseteq \tilde{\Phi}(V^n)$ for each $n \in \omega$. For each $n \in \omega$ let $B^n = B(V^n)$, and let $\tilde{B}^n = \tilde{\Phi}(V^n)$. As before, let $C = \bigcap \{V^n : n \in \omega\} = \bigcap \{B^n : n \in \omega\} = \bigcap \{\tilde{B}^n : n \in \omega\} = \Pi\{C_\alpha : \alpha \in \kappa\}$. Finally, let $N = \{n \in \omega : \text{supp}(\tilde{B}^n) = \varnothing\}$; either $N = \omega$, or $|N| < \omega$. If $N = \omega$, the choice of Φ clearly ensures that $C \neq \varnothing$. If N is finite, we may argue just as in the proof of 4.4 that again $C \neq \varnothing$. □

4.7. THEOREM. *If X is weakly α-favorable, so is* $S^\kappa X$.

Proof. Either follow the pattern of 4.6, or (even easier) define the functions $\tilde{\Phi}_n (n \geq 1)$ as follows.

As before, it suffices to define $\tilde{\Phi}_n$ on ${}^n\mathfrak{B}$. Suppose that $\langle B^0, \ldots, B^{n-1} \rangle \in {}^n\mathfrak{B}$ is compatible. For $1 \leq i \leq n-1$ let

$$I_i = \{\alpha \in \text{supp}(B^i) : \pi_\alpha[\tilde{\Phi}_i(\langle B^0, \ldots, B^{i-1} \rangle)] = \Phi_i(\langle B^0_\alpha, \ldots, B^{i-1}_\alpha \rangle)\},$$

and suppose that each I_i is finite and contains $\bigcup\{\operatorname{supp}(B^j): j < i\}$. Let $I = \bigcup\{I_i: 1 \leq i \leq n-1\} \cup \operatorname{supp}(B^{n-1})$, let

$$\tilde{B}_\alpha = \begin{cases} \Phi_n(\langle B_\alpha^0, \ldots, B_\alpha^{n-1}\rangle), & \text{if } \alpha \in I \\ \bigcup_{\beta \in I} \tilde{B}_\beta, & \text{otherwise,} \end{cases}$$

and put $\tilde{\Phi}_n(\langle B^0, \ldots, B^{n-1}\rangle) = \tilde{B}$. □

All the α-favorability properties are inherited by dense G_δ-subspaces. ([**1**, **8**])

4.8. *Question*. When is $\mathbf{C}^\kappa X$ (or $\mathbf{F}^\kappa X$ or $\Phi^\kappa X$) a G_δ-set in $\mathbf{S}^\kappa X$? More generally, when can $\mathbf{S}^\kappa X$ in 4.4, 4.6, and 4.7 be replaced by one of these dense subspaces? (Clearly we can replace $\mathbf{S}^\kappa X$ by $\mathbf{C}^\kappa X$ whenever $\sigma X \supseteq [X]^\omega$, as may be seen by examining the proofs.)

4.9. THEOREM. *Suppose that X is pseudo-complete. Then $\mathbf{S}^\kappa X$ is pseudo-complete; moreover, $\mathbf{C}^\kappa X$ is pseudo-complete if $\kappa \geq |X|$.*

Proof. Let $\tilde{X} = \mathbf{S}^\kappa X$; clearly $\tilde{X}$ is π-regular. Let $\langle \mathfrak{B}_n: n \in \omega\rangle$ be a sequence of π-bases witnessing the pseudo-completeness of X. For each $n \in \omega$ let $\tilde{\mathfrak{B}}_n$ be the family of all $B \in \tilde{\mathfrak{B}}$ such that either

(a) $\operatorname{supp}(B) = \varnothing$ and $B_0 \in \mathfrak{B}_n$ or

(b) (i) $B_\alpha \in \mathfrak{B}_n$ for every $\alpha \in \operatorname{supp}(B)$;

(ii) for any $\alpha, \beta \in \operatorname{supp}(B)$, either $B_\alpha = B_\beta$, or $B_\alpha \cap B_\beta = \varnothing$; and

(iii) $B_\alpha \cap B_\beta = \varnothing$ for some $\alpha, \beta \in \operatorname{supp}(B)$.

(Compare (b) with the construction in the proof of 4.6.) Each $\tilde{\mathfrak{B}}_n$ is then a π-base for $\tilde{X}$.

Now suppose that $\langle B^n: n \in \omega\rangle$ is such that $\operatorname{cl}_{\tilde{X}} B^{n+1} \subseteq B^n \in \tilde{\mathfrak{B}}_n$ for each $n \in \omega$, and let $C = \bigcap\{B^n: n \in \omega\} = \bigcap\{\operatorname{cl}_{\tilde{X}} B^n: n \in \omega\}$. For $\alpha \in \kappa$ let $C_\alpha = \pi_\alpha[C] = \bigcap\{B_\alpha^n: n \in \omega\}$; we must show that each C_α is non-empty. If $\operatorname{supp}(B^n) = \varnothing$ for all n this is obvious: for each $\alpha \in \kappa$, $C_\alpha = C_0 = \bigcap\{B_0^n: n \in \omega\} \neq \varnothing$ by the pseudo-completeness of X. Otherwise let $I = \bigcup\{\operatorname{supp}(B^n): n \in \omega\} \neq \varnothing$, and note that conditons (b)(ii) and (b)(iii) ensure that $\operatorname{supp}(B^n) \subseteq \operatorname{supp}(B^{n+1})$ for each n. For each $\alpha \in I$ there is therefore a $k_\alpha \in \omega$ such that $\alpha \in \operatorname{supp}(B^n)$ for each $n \geq k_\alpha$; but then $B_\alpha^n \in \mathfrak{B}_n$ for each $n \geq k_\alpha$, whence $C_\alpha = \bigcap\{B_\alpha^n: n \geq k_\alpha\} \neq \varnothing$. And if $\alpha \in \kappa \setminus I$, then $B_\alpha^n \supseteq B_\gamma^n$ for each $n \in \omega$ and $\gamma \in I$, so $C_\alpha \supseteq \bigcup\{C_\gamma: \gamma \in I\} \neq \varnothing$. Thus, $C \neq \varnothing$, and $\tilde{X}$ is pseudo-complete.

Now assume that $\kappa \geq |X|$; to prove that $\mathbf{C}^\kappa X$ is pseudo-complete, it suffices to show that $C \cap \mathbf{C}^\kappa X \neq \varnothing$, where C is as in the preceding paragraph. For each $\alpha \in I$ choose $p_\alpha \in C_\alpha$ arbitrarily, and let $K = \mathrm{cl}_X\{p_\alpha: \alpha \in I\}$; if, now, $\gamma \in \kappa \setminus I$, then $K \subseteq C_\gamma$, since C_γ is a closed set containing each p_α ($\alpha \in I$). Assume for the moment that

$$|\kappa \setminus I| \geq |K \setminus \{p_\alpha: \alpha \in I\}|.$$

Then there is a surjection $f: \kappa \setminus I \to K \setminus \{p_\alpha: \alpha \in I\}$, and the point x defined by

$$x_\alpha = \begin{cases} p_\alpha, & \text{if } \alpha \in I \\ f(\alpha), & \text{if } \alpha \in \kappa \setminus I \end{cases}$$

is in $C \cap \mathbf{C}^\kappa X$.

The remaining possibility is that $|\kappa \setminus I| < |k \setminus \{p_\alpha: \alpha \in I\}|$, which can occur only when $\kappa = \omega$. Then $|X| \leq \omega$. Now, a countable space containing a dense-in-itself, open subset is easily seen *not* to be pseudo-complete: a decreasing sequence of π-basic open sets can be found that 'squeezes out' each point of X. Thus, in this case X must have a dense set, D, of isolated points, and we may assume that $\mathfrak{B}_n = \{\{x\}: x \in D\}$ for each $n \in \omega$. We then modify the definition of $\tilde{\mathfrak{B}}_n$ slightly by adding a fourth condition to clause (b).

(b) (iv) $B_i \in \mathfrak{B}_n$ for each $i \leq n$.

It then follows that either I is finite, in which case clearly $\Phi^\omega X \cap C \neq \varnothing$, or $I = \omega$ and $C = \{x\}$ for some $x \in \mathbf{F}^\omega X$. In either case $C \cap \mathbf{C}^\omega X \neq \varnothing$, so $\mathbf{C}^\omega X$ is pseudo-compact. □

4.10. Remark. The condition that κ be at least $|X|$ is clearly stronger than necessary: the proof of the last part of 4.9 requires only that κ be at least $\sup\{|Y|: Y \subseteq X$ and Y is separable$\}$, except possibly when this number is ω (i.e., when X is $\aleph_0$-bounded) and $|X| > \omega$.

The last part of the proof of 4.9 yields the following corollary.

4.11. Corollary. *If X has a dense set of isolated points, then $\mathbf{C}^\omega X$ is pseudo-complete.* □

5. Hyperspaces.

If $\kappa = |X|$, the hyperspace of X, 2^X, may be thought of as a sort of 'unordered' version of $\mathbf{C}^\kappa X$. (See [5] for background information on 2^X.) In a sense, 2^X is to $\mathbf{C}^\kappa X$ as $[X]^2$ is to ${}^2X$. Let us make this more precise.

5.0 Definition. Let X be a topological space. The *hyperspace* of X, denoted by 2^X, is the set $\sigma X \setminus \{\varnothing\}$ of non-empty closed subsets of X endowed with the Vietoris topology, i.e., that given by the base $\mathcal{K}(X)$

consisting of all sets of the form $((V_0,\dots,V_{n-1}))$, where $n \geq 1$, $V_i \in \tau X \setminus \{\varnothing\}$ for $i < n$, and $((V_0,\dots,V_{n-1})) = \{C \in 2^X\colon C \subseteq \bigcup\{V_i\colon i<n\}$ and $\forall i < n(C \cap V_i \neq \varnothing)\}$.

5.1. THEOREM. *Let* $P\colon \mathbf{C}^\kappa X \to [X]^{\leq\kappa} \cap 2^X\colon x \mapsto \operatorname{ran} x$. *Then* P *is an open, continuous surjection.*

Proof. P is clearly a surjection. For $((V_0,\dots,V_{n-1})) \in \mathcal{K}(X)$, let

$$((V_0,\dots,V_{n-1}))^\kappa = ((V_0,\dots,V_{n-1})) \cap [X]^{\leq\kappa};$$

the collection of all such sets is a base for $2^X \cap [X]^{\leq\kappa}$.

Let $B(F,V)$ be a basic open set in $\mathbf{C}^\kappa X$. If $F = \{\alpha_0,\dots,\alpha_{n-1}\}$, say, let $W_i = V_{\alpha_i}$ for $i<n$. Then it is easy to see that $P[B(F,V)] = ((W_0,\dots,W_{n-1}))^\kappa$, so P is open.

Now let $((V_0,\dots,V_{n-1}))^\kappa$ be a basic open set in $2^X \cap [X]^{\leq\kappa}$. For each $\phi \in {}^n\kappa$ let $W_\phi = B(\operatorname{ran}\phi, V\circ\phi^{-1})$, where $V\colon n \to \tau X\colon i \mapsto V_i$. Clearly $W = \bigcup\{W_\phi\colon \phi \in {}^n\kappa\}$ is open in $\mathbf{C}^\kappa X$; and since obviously $P[W_\phi] = ((V_0,\dots,V_{n-1}))^\kappa$ for each $\phi \in {}^n\kappa$, $P[W] = ((V_0,\dots,V_{n-1}))^\kappa$ as well. On the other hand, if $x \in \mathbf{C}^\kappa X \setminus W$, then either $V_i \cap \operatorname{ran} x = \varnothing$ for some $i<n$, or $\operatorname{ran} x \nsubseteq \bigcup\{V_i\colon i<n\}$. In either case $P(x) \notin ((V_0,\dots,V_{n-1}))^\kappa$, so $W = P^{-1}[((V_0,\dots,V_{n-1}))^\kappa]$, and P is continuous. □

5.2. COROLLARY. *For any* $\kappa \geq \omega$, *if* $\mathbf{C}^\kappa X$ *has one of the following properties*, 2^X *has the same property.*

(i) *Baireness*
(ii) *weak α-favorability*
(iii) *α-favorability*

Proof. Properties (i)–(iii) are preserved by open, continuous surjections. (For (i) this is well known. For (ii) it is proved in [8]; the proof for (iii) is even easier.) It is easy to see that if a space Y has a dense subspace with one of these properties, then Y has that property as well. Thus, the result follows immediately from 5.1 and the observation that $[X]^{\leq\kappa} \cap 2^X$ is dense in 2^X. □

We do not know whether there are corresponding results for pseudo-completeness or strong α-favorability. Although a space with a dense pseudo-complete subspace is pseudo-complete, pseudo-completeness is not known to be preserved by open, continuous surjections. Strong α-favorability, on the other hand, does not transfer up from dense subspaces. [Let $\mathbf{Q}$ be the usual space of rational numbers. Let X denote the space $\mathbf{Q} \times \mathbf{Q}$ with the finer topology obtained by isolating each point $\langle p, q \rangle$ with $q \neq 0$. Then X has a dense set of isolated points, which is obviously a strongly α-favorable subspace, but it is not hard to show that X itself is not strongly α-favorable. Just enumerate $\mathbf{Q}$ as $\{q_n : n \in \omega\}$, and suppose that Φ is a strong winning strategy on X. Let $x_0 = \langle q_1, 0 \rangle$, $V_0 = \{y \in X: \|y - x_0\| < 1\}$, where $\|\cdot\|$ is the Euclidean norm. Given $\Phi(x_n, V_n)$ for some $n \in \omega$, let

$$k = \min\{i \in \omega: i > n \text{ and } \langle q_i, 0 \rangle \in \Phi(x_n, V_n)\},$$

let $x_{n+1} = \langle q_k, 0 \rangle$, and let

$$V_{n+1} = \{y \in \Phi(x_n, V_n): \|y - x_{n+1}\| < 2^{-n-}\} \setminus \{\langle q_i, 0 \rangle : i \leq n\}.$$

Clearly $\bigcap \{V_n : n \in \omega\} = \varnothing$.]

However, McCoy has shown (4) that 2^X is pseudo-complete if X is.

5.3. COROLLARY. *If X is pseudo-complete, and $\kappa \geq |X|$, then both $C^\kappa X$ and 2^X are pseudo-complete.* □

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