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ON PLANAR APPROXIMATIONS AND CONVERGENCE

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Abstract

In [12] the authors introduced planar approximations in their study of separately continuous functions $f : [0, 1] \times [0, 1] \to \mathbb{R}$. In Section 2 of this paper we show how, when dealing with continuity in the ordinary sense, the convergence which was pointwise for separately continuous functions, becomes uniform. In Section 3, we look at sequences of planar approximable functions and their limits under various notions of convergence.

1. Introduction

For a function $f : [0, 1] \times [0, 1] \to \mathbb{R}$, the function $f_x : [0, 1] \to \mathbb{R}$, given by $f_x(y) = f(x, y)$, where x is fixed is called an x-section of f. Similarly we can define the y-section of f. A function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is called separately continuous if each x-section and y-section is a continuous

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function. This is *not* the same as continuity in the ordinary sense (referred to as joint continuity) with the first counterexample appearing in the literature in 1873. This example is

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$$

In [12], the above authors studied separately continuous functions, with one result involving planar approximations (which are defined below) of a separately continuous f. In that paper we showed that if f is separately continuous, then the planar approximations converge pointwise to f. We also gave examples showing that some, but not all, Baire class one functions are planar approximable, and that there are symmetrically quasi-continuous functions [10] which are not planar approximable.

We begin by defining these approximations.

Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$. We define $P_n(x, y)$, the planar approximations to f as follows: for $n = 0$, we start with the unit square and divide it into two triangles by splitting it along the diagonal joining (1, 0) and (0, 1). So our first triangle has corners (0, 0), (1, 0), and (0, 1) while the second triangle has corners (1, 0), (0, 1), and (1, 1). For each triangle, we find the image of the corner points and, using the triples $(x, y, f(x, y))$, we create a planar region through these triples. Adjoining the two planar regions, we obtain our first planar approximation, $P_0(x, y)$.

At stage n, divide the unit square into 2^n subsquares of side length $1/2^n$. Then divide each square into two triangles for a total of 2^{n+1} triangles. So each triangle has vertices $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) and we use $(x_i, y_i, f(x_i, y_i)), i = 1, 2, 3$ to create a section of a plane in $\mathbb{R}^3$. Joining these sections together give $P_n(x, y)$, where for a given (x, y), $P_n(x, y)$ is the z-value of the plane section above that point. Let T_n denote all the corner points for the triangles at state n and let $T = \bigcup T_n$. We note here that for $k < n$ we have $T_k \subset T_n$, for a fixed

(x, y) the set of corners used in approximating $f(x, y)$ converges to (x, y), and if $(x, y) \in T_k$, then $P_n(x, y) = f(x, y)$ for $n \geq k$.

2. Continuous Functions and Planar Approximations

The purpose of this note is to further delineate between separately and jointly continuous functions by looking at them in terms of these planar approximations. We shall show that for jointly continuous functions the approximations actually converge uniformly and give an example of a separately continuous function where the convergence must be pointwise.

Theorem 1. *If* $f : [0, 1] \times [0, 1] \to \mathbb{R}$ *is continuous, then the planar approximations* P_n *converge uniformly to* f.

Proof. Since $[0, 1] \times [0, 1]$ is compact, the function f is actually uniformly continuous. Thus for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $d((x_1, y_1), (x_2, y_2)) < \delta$ implies $|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon/5$, where d is the Euclidean metric on the unit square. Now pick $N \in \mathbb{N}$ such that $\frac{\sqrt{2}}{2^N} < \delta$. Then for every $(x, y) \in [0, 1] \times [0, 1]$ the image of the corner points in T_N which form the plane to approximate $f(x, y)$ are within $\frac{\varepsilon}{5}$ of $f(x, y)$. So we have

$$|f(x, y) - P_N(x, y)| \leq \frac{\varepsilon}{5}.$$

However, this N is fixed. Notice that for this fixed N and for each planar section, the image of the corners which form the section are at most $\frac{2\varepsilon}{5}$ apart leading us to $|P_N(x_1, y_1) - P_N(x_2, y_2)| \leq \frac{2\varepsilon}{5}$ for any (x_1, y_1) and (x_2, y_2) in the triangle which defines the planar section.

To get our result, we again refer to the uniform continuity of f. Then for $n, m \geq N$ we know $|P_n(x, y) - P_m(x, y)| \leq \frac{2\varepsilon}{5}$ since the points that form both P_n and P_m above (x, y) are within $\frac{\varepsilon}{5}$ of $f(x, y)$. So letting

$(x, y) \in [0, 1] \times [0, 1]$ and letting $(a, b) \in T_N$ be a corner used in finding $P_N(x, y)$ we have for $n > N$

$$|f(x, y) - P_n(x, y)| \le |f(x, y) - f(a, b)| + |P_N(a, b) - P_N(x, y)| + |P_N(x, y) - P_n(x, y)| < \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5} = \varepsilon.$$

Since, (x, y) was arbitrary we have shown the uniform convergence of the P_n. □

As the following example shows, this is not true for functions that are *separately continuous*, but not jointly continuous. The F below is planar approximable (since it is separately continuous), but the convergence of the P_n is pointwise.

Example 1. Let $F : [0, 1] \times [0, 1] \to \mathbb{R}$ be given by

$$F(x, y) = \begin{cases} \dfrac{1}{x + y} & x = y,\ xy > 0 \\ \dfrac{x}{2y^2} & 0 \le x < y \\ \dfrac{y}{2x^2} & 0 \le y < x \\ 0 & (x, y) = (0, 0). \end{cases}$$

3. Convergence of Planar Approximable Functions

The main question is this: If $\{f_n\}$ is a sequence of planar approximable functions from $[0, 1]^2$ into $\mathbb{R}$ and f_n converges with respect to some type of non-standard convergence to some function f, then is f also planar approximable?

3.1. Convergence on dense sets

We say that a sequence f_n of functions from $[0, 1]^2$ into $\mathbb{R}$ *converges on a dense set to* f, when

$$\{(x, y) \in [0, 1]^2 : f_n(x, y) \to f(x, y)\} \text{ is dense in } [0, 1]^2.$$

So, does every sequence f_n of planar approximable functions converge on a dense set to a planar approximable function? The answer is *no*.

In fact, pick (x_0, y_0) which is not a corner point (not in the set T) and let $\{(x_n, y_n)\}$ be a sequence in T which converges to (x_0, y_0). Let

$$f_n(x, y) = \chi_{(x_n, y_n)}(x, y)$$

and

$$f(x, y) = \chi_{(x_0, y_0)}(x, y).$$

Each f_n is planar approximable, f_n converges to f on a dense set, yet f is not planar approximable.

3.2. Continuous convergence

We say f_n *converges continuously* to f if for all x_0 and all sequences $\{x_n\}$ which converges to x_0 we have

$$f_n(x_n) \to f(x_0).$$

This condition has a long history in mathematical literature. Actually, Sierpiński [14] refers to the following theorem as

Kuratowski Theorem. Let X be a compact metric space. A necessary and sufficient condition for a sequence $\{f_n\}$ of continuous functions converges uniformly to f is that for every point x_0 in X and any sequence x_n in X such that $\lim x_n = x_0$ implies $\lim f_n(x_n) = f(x_0)$.

We will prove now that continuous convergence preserves planar approximable functions.

Let $\varepsilon > 0$ and P_n be a planar approximation of f. For any point (x, y) there is a sequence of triangles whose closure contains (x, y) which are decreasing down to the point. Make three sequences $\{(s_n, t_n)\}$, $\{(s_n', t_n')\}$, and $\{(s_n'', t_n'')\}$, where these are the corners of the n-th triangle.

From the definition, there is an N such that $n > N$ implies $f_n(s_n, t_n)$, $f_n(s'_n, t'_n)$, and $f_n(s''_n, t''_n)$ are $\varepsilon/4$ close to $f(x, y)$. In a similar vein, if n is large enough

$$|f_n(s_n, t_n) - f(s_n, t_n)| \leq \frac{\varepsilon}{4}.$$

This means that the corners that make up the triangle have $P_n(x, y)$ at most $2 \cdot \frac{\varepsilon}{4}$ away from any corner. Thus

$$|P_n(x, y) - f(x, y)| \leq |P_n(x, y) - f_n(s_n, t_n)| + |f_n(s_n, t_n) - f(x \cdot y)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.$$

Hence f is planar approximable.

We note here that our proof works for any domain $D \subseteq \mathbb{R} \times \mathbb{R}$, but that there is simpler reasoning for the unit square. If the domain is a compact set, then continuous convergence for a sequence of functions is equivalent to uniform convergence, see Kuratowski's theorem above.

Now, because $x_n = x$ is a sequence converging to x continuous convergence implies pointwise convergence. *It does not work the other way around*. If

$$g_n(x) = \begin{cases} 1/n & x \neq 0 \\ 1 & x = 0, \end{cases}$$

then g_n converges pointwise to $\chi_{\{0\}}$, but does not converge to it continuously. So if we can get our result working with pointwise convergence this would imply the continuous convergence result. However, that's not going to work. That is,

> There is f_n each of which is planar approximable with $f_n \to f$ pointwise, but f is not planar approximable.

Pick a point $(x_0, y_0) \notin T$ and let $\{\tau_n\}$ be the sequence of the closed triangles (including the interior, not just the sides) converging down to

(x_0, y_0). Define f_n by

$$f_n((x, y)) = \begin{cases} 1/n & (x, y) \notin \tau_n \\ 1 & (x, y) \in \tau_n. \end{cases}$$

This is really just a variation on the example above it. The limit is not planar approximable because $P_n \equiv 0$ (see discrete convergence).

3.3. Almost uniform convergence

For the basic results and definitions (see [4], [8]). It is well known that a sequence of monotonic functions defined on an open interval I converges to a continuous function f, then the sequence converges almost uniformly to f; that is, it converges uniformly to f on any compact subset K of I. In our situation, the role of I is the *open* unit square. The only place this can "mess up" is on the boundary. Pick a point x so that $(0, x)$ is not in T. Then there are points $t_n < x < s_n$, where $s_n, t_n \to x$ and $(0, s_n)$, $(0, t_n) \in T$. Let

$$f_n(x, y) = \begin{cases} 1 & (x, y) \in \{0\} \times [t_n, s_n] \\ 0 & \text{otherwise.} \end{cases}$$

Each f_n is planar approximable, f_n converges to $\chi_{(0, x)}$ pointwise (and in a decreasing fashion), and, in the interior of the unit square, $f_n \equiv 0$ so f_n converges to f almost uniformly. The limit f is not planar approximable since f is zero at every point in T.

3.4. Graph convergence

Let us look at the following classical result, see [7] or ([16] where this result is repeated):

Theorem. *Let X and Y be compact metric spaces. Let $f_1, f_2, \ldots$ be functions from X to Y with graphs $\Gamma(f_1), \Gamma(f_2), \ldots$ in $X \times Y$. Then $\lim \Gamma(f_n)$ exists and is the graph of a function f if and only if f_n converges to f and f is continuous.*

It is the "if" condition, in terms of the sequence of the graphs $\Gamma(f_n)$ that provides the definition of *graph convergence* of a sequence of functions.

We shall now prove that graph convergence *preserves* planar approximability.

In fact, let $\varepsilon > 0$ and pick an arbitrary point (x, y). For this ε there exists a positive integer N such that if $n > N$ the open ball about $\Gamma(f)$ with radius $\varepsilon/3$ (in the sup norm). Fix such an f_n. This f_n is planar approximable so we know that $P_{n,m}$ converges pointwise to f_n. Thus there exists an $M > N$ so that for $m > M$

$$|P_{n,m}(x, y) - f_n(x, y)| < \varepsilon/3.$$

Since all the points in f_n are within $\varepsilon/3$ of f, the three corners which make the triangle containing (x, y) have their images under f_n within $\varepsilon/3$ of their images under f. Hence the planar approximations $P_{n,m}$ and P_m of the triangle using f_n and f are $\varepsilon/3$ close for every point in the triangle. Thus

$$\begin{aligned} |P_m(x, y) - f(x, y)| &\le |P_m(x, y) - P_{n,m}(x, y)| + |P_{n,m}(x, y) - f_n(x, y)| \\ &\quad + |f_n(x, y) - f(x, y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

So P_m converges pointwise to f which means f is planar approximable.

3.5. Quasi-normal convergence

Let $f_1, f_2, \ldots$ be real-valued functions defined on a set X. We say that the sequence $\{f_n\}$ *converges quasi-normally to* f on X, if there is a sequence $\{\varepsilon_n\}$ of nonnegative reals converging to zero such that for every $x \in X$ there is an index k_x such that

$$|f_n(x) - f(x)| \le \varepsilon_n \text{ for every } n \ge k_x.$$

The studies of quasi-normal convergence were initiated by Čsaszar and

Laczkovich [2, 3]. We shall prove that quasi-normal convergence *does not* preserve planar approximability. In fact, let (x_0, y_0) be a point not in T. For this point (x_0, y_0) and at each step n, there is a triangle with corners (s_n, t_n), (s'_n, t'_n), and (s''_n, t''_n) containing (x_0, y_0) in its interior. Define f_n as 1 in the closure of this triangle and 0 elsewhere. This f_n is obviously planar approximable and converges quasi-normally to $f = \chi_{(x_0, y_0)}$ (after step n everything outside of the triangle agrees with its f value and $f(x_0, y_0) = f_n(x_0, y_0) = 1$ always). However, f is not planar approximable.

3.6. Discrete convergence

A variation of the above example also works for discrete convergence. Recall that a sequence $f_n : \mathbb{R} \to \mathbb{R}$ *discretely converges to the limit* f (see [2, 3]) if

$$\forall x \, \exists n\,(x) \, \forall n > n(x) \; f_n(x) = f(x).$$

This is between pointwise and uniform convergence. Pick $(x_0, y_0) \notin T$ and let

$$f_n((x, y)) = \begin{cases} 1 & (x, y) \notin \tau_n \\ 0 & (x, y) \in \tau_n, \end{cases}$$

where τ_n is the triangle at stage n that contains (x_0, y_0). This converges discretely to $\chi_{\{(x, y)\}}$ which is not planar approximable.

The studies of discrete convergence were initiated by Čsaszar and Laczkovich [2, 3]. For other types of convergence, such as quasi-uniform convergence, see [1, p. 406], [6], [13], and [9, p. 255].

Lastly, we wish to end this paper with an open question. The question is not new, having appeared in our previous paper, but is still unanswered.

Problem 3. Does there exist a complete description of the functions which can be approximated pointwise by these P_n ?