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PREIMAGES OF BAIRE SPACES - AN EXAMPLE

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Abstract. An answer, in the negative, is provided to a question of Noll as to whether Baire spaces are preserved under inverse images of functions that are feebly open, feebly continuous and fibre-complete.

Keywords: Baire spaces, preimage, feebly open, feebly continuous, fibre-complete.

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Let us start with some definitions which are due to Noll [2] . Given a topological space X, a pair (φ, T) consisting of a tree T of height $\aleph_0$ and a mapping φ with domain T is called a *web* on X if the following conditions are met:

(i) $\{\varphi(t) : t \in T\}$ is a π-base for X;

(ii) for fixed $t \in T$, the set $\{\varphi(s) : t <_T s \in T\}$ is a π-base for $\varphi(t)$.

Recall that a family B of open subsets of a space X is called a *π-base* for X if every nonempty open U in X contains some nonempty $V \in B$, see Oxtoby [3].

Let $f : X \to Y$ be a function. Then f is called *fibre-complete* if there exists a web (φ, T) on X such that for every $y \in Y$ and every cofinal branch (t_n) in T (i.e. $t_n <_T t_{n+1}$ for all n) having $\varphi(t_n) \cap f^{-1}(y) \neq \phi$ for every n, the intersection $\cap\{\varphi(t_n) : n \in \mathbb{N}\}$ is nonempty.

Let X be a regular space and suppose that $f : X \to Y$ has countably compact or sequentially compact or pseudo-compact fibres. Then f is fibre-complete, see [2, (1), p.848].

Following Frolik [1], we say that a function $f : X \to f(X)$ is *feebly open* (resp. *feebly continuous*) if the image (resp. inverse image) of a nonempty open set has a nonempty interior. Noll [2, (2), p.850] posed the following open question.

Question. Is the pre-image of a Baire space under a feebly open, feebly continuous and fibre-complete function, a Baire space?

The following Example answers this question in the negative.

Example. Enumerate the set of rational numbers in $(-1, 0)$ as the set $P = \{q_1, q_2, \dots\}$. Consider a countable family of closed intervals in $\mathbb{R}$ each of length 2/3: $I_1 = [1 - \frac{1}{3}, 1 + \frac{1}{3}], I_2 = [2 - \frac{1}{3}, 2 + \frac{1}{3}], \dots, I_n = [n - \frac{1}{3}, n + \frac{1}{3}]$, for each $n \in \mathbb{N}$. Let $F = \cup\{[n - \frac{1}{3}, n + \frac{1}{3}] : n \in \mathbb{N}\}$. Let $P \cup F$ and $\mathbb{N}$ have subspace topologies induced by the usual topology on

$\mathbb{R}$. Now, we define $f : P \cup F \to \mathbb{N}$ by $f(x) = n$, if $x \in \{q_n\} \cup I_n, n = 1, 2, 3, \ldots$. The function f is open (hence it is feebly open), feebly continuous and has compact fibres $f^{-1}(n), n \in \mathbb{N}$, so that f is fibre-complete.

Now the range of f (the space $\mathbb{N}$ of natural numbers) is Baire, while the domain space is not Baire since $\mathbb{Q} \cap (-1, 0)$ is of the first category, where $\mathbb{Q}$ denotes the set of rational numbers.

References

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