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Quasi-continuity and product spaces

by

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At the beginning we recall the definitions of notions used in this paper.

DEFINITION 1. A space will be called *quasi-regular* if for every nonempty open set U, there is a nonempty open set V such that $\bar{V} \subset U$ (see [5], p. 39 and [8], p. 164).

DEFINITION 2. A function $f\colon X \to Y$ is called *quasi-continuous at a point* $x \in X$ if for each open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$, we have $A \cap \operatorname{Int} f^{-1}(H) \neq \emptyset$. A function $f\colon X \to Y$ is called *quasi-continuous*, if it is quasi-continuous at each point x of X ([5], p. 39, compare [4] p. 186).

Now we introduce the definition of symmetrically quasi-continuous function.

DEFINITION 3. A function $f\colon X \times Y \to Z$ (X, Y, Z — arbitrary topological spaces) is said to be *quasi-continuous at* $(p, q) \in X \times Y$ *with respect to the variable* y (compare [5], p. 41 also [4], p. 188), if for every neighborhood N of $f(p, q)$ and for every neighborhood $U \times V$ of (p, q), there exists a neighborhood V' of q, with $V' \subset V$, and a non-empty open $U' \subset U$, such that for all $(x, y) \in U' \times V'$ we have $f(x, y) \in N$. If f is quasi-continuous with respect to the variable y at each point of its domain, it will be called *quasi-continuous with respect to* y. The definition of a function f that is quasi-continuous with respect to x is quite similar. If f is quasi-continuous with respect to x and y, we will say that f is *symmetrically quasi-continuous*.

Let X, Y and Z be spaces and let a function $f\colon X \times Y \to Z$ be given. For every fixed $x \in X$, the function $f_x\colon Y \to Z$ defined by $f_x(y) = f(x, y)$, where $y \in Y$, is called an *x-section of* f. An y-section of f is defined similarly.

One can easily show from the definitions that if f is symmetrically quasi-continuous, then f_x and f_y are quasi-continuous for all $x \in X$ and $y \in Y$. The converse does not hold.

EXAMPLE 1. Indeed, define $f\colon I_0 \times I_0 \to I_0$ (I_0 denotes the segment $\langle 0, 1\rangle$) as follows: $f(x, y) = 0$ if $x \in \langle \frac{1}{2}, 1\rangle$ and $y \in \langle 0, \frac{1}{2}\rangle$ and $f(x, y) = 1$ on the rest. It is easy to verify that all x-sections f_x and all y-sections f_y of f are quasi-continuous and f is not symmetrically quasi-continuous. However, we have the following

THEOREM 1. *Let X be a Baire space, Y be first countable and Z be regular. If f is a function on $X \times Y$ to Z such that all its x-sections f_x are continuous and all its y-sections f_y are quasi-continuous, then f is quasi-continuous with respect to y.*

So, we obtain at once

COROLLARY 1. *Let X and Y be first countable, Baire spaces and Z be a regular one. If $f: X \times Y \to Z$ has all its x-sections and y-sections continuous, then f is symmetrically quasi-continuous.*

We note here that some of the statements of this paper were inspired by [5] and [6].

Proof of Theorem 1. Suppose that f is not quasi-continuous with respect to y. The proof is divided into three parts. We sketch them:

Part 1. We conclude some consequences from the fact that f is not quasi-continuous with respect to y. Some special sets are defined. Namely: There is a point (x_0, y_0) and its neighborhood $U \times V$ and a neighborhood N of $f(x_0, y_0)$ such that, if V' is a neighborhood of y_0 with $V' \subset V$ and U' is an open, nonempty set, $U' \subset U$, then we have

$$(*) \qquad f(U' \times V') \cap (Z \backslash N) \neq \emptyset.$$

Let V_n be a local countable basis at y_0, say, contained in V. Let $N_0 \subset Z$ be an open set, such that

$$f(x_0, y_0) \in N_0 \subset \bar{N}_0 \subset N.$$

By quasi-continuity of f_{y_0} at x_0 we have

$$W = \operatorname{Int} f_{y_0}^{-1}(N_0) \cap U \neq \emptyset.$$

Now we define the sets A_n as follows:

$$A_n = \{x: x \in W, V_n \subset f_x^{-1}(N_0)\}.$$

Part 2. We show that $W = \bigcup_{n=1}^{\infty} A_n$. The inclusion $\bigcup_{n=1}^{\infty} A_n \subset W$ is obvious. If $x \in T$, then $x \in W$, hence $f_{y_0}(x) \in N_0$. So, $f_x(y_0) \in N_0$ and $y_0 \in f_x^{-1}(N_0) \cap V$, but $f_x^{-1}(N_0) = \operatorname{Int} f_x^{-1}(N_0)$, by continuity of f_x. Thus there is $n \in \mathcal{N}$ (the set of all natural numbers), such that $V_n \subset f_x^{-1}(N_0)$.

Part 3. We show that for each n, the set A_n is nowhere dense in W. Let $S \subset U$ be an arbitrary nonempty set. Let us form $S \times V_n$ for given n. Because of $(*)$ from Part 1, there exists $(x_1, y_1) \in S \times V_n$ such that $f(x_1, y_1) \notin N$. We choose a neighborhood N_1 of $f(x_1, y_1)$ such that $N_0 \cap N_1 = \emptyset$. Using quasi-continuity of f_{y_1} at x_1 we obtain that there exists a nonempty open set $S_1 \subset S$ such that $f(x, y_1) \in N_1$ for any $x \in S_1$, hence $f(x, y_1) \notin N_0$. Thus $y_1 \notin f_x^{-1}(N_0)$ for any $x \in S_1$. This implies $V_n \not\subset f_x^{-1}(N_0)$ hence $x \notin A_n$. Thus $S_1 \cap A_n = \emptyset$. This shows that A_n is nowhere dense and finishes Part 3.

Now we obtain that the open set $W = \bigcup_{n=1}^{\infty} A_n$ is of the first category. This is a contradiction and the proof of Theorem 1 is finished.

Remark 1. The assumption of quasi-continuity of f_y cannot be weakened. Recall that a function $f: X \to Y$ is called *somewhat continuous* ([3], p. 6) if for each set $V \subset Y$, open in Y, such that $f^{-1}(V) \neq \emptyset$ there exists an open set $U \subset X$, $U \neq \emptyset$, with $U \subset f^{-1}(V)$. There is a counterexample that using somewhat continuity instead of quasi-continuity, Theorem 1 becomes false. Indeed, define $f: I_0 \times I_0 \to I_0$ by: $f(x, y) = 0$ if $x \in \langle 0, \frac{1}{2} \rangle$ or $x \langle \frac{1}{2}, 1 \rangle$ and x is rational; $f(x, y) = 1$ if $x \in (\frac{1}{2}, \frac{1}{2} \rangle$ or $x \in (\frac{1}{2}, 1 \rangle$ and x is irrational. Such a function has all its x-sections continuous and has all its y-sections somewhat continuous and Theorem 1 does not hold for f.

Also the assumption of continuity of all x-sections f_x of f is essential – see Example 1.

It follows from [7] and Corollary 1, that if X and Y are second countable, Baire spaces and Z is a regular one, and a function $f: X \times Y \to Z$, then the following implications hold (which show the inclusion relations between proper classes of functions) – see Diagram 1. None of these implications can, in general, be replaced by an equivalence. The examples showing this may be found in [6], [7] – see also Example 1 and Remark 1.

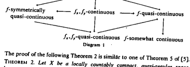

Diagram 1

The proof of the following Theorem 2 is similar to one of Theorem 5 of [5].

THEOREM 2. *Let X be a locally countably compact, quasi-regular space, Y be a topological one, and Z be metric. If f is a function on $X \times Y$ to Z which is quasi-continuous with respect to y, then the set of points of continuity of f contains a subset of $X \times \{p\}$ for all $p \in Y$.*

Now we will prove the following

LEMMA 1. *Every quasi-regular, locally countably compact space X is a Baire space.*

Proof. Let $\{D_i\}$ be a sequence of dense, open subsets of X, and let U_1 be any nonempty, open subset of X. Since X is locally countably compact and quasi-regular, there exists a nonempty set U_2, open in X, whose closure is countably compact and contained in $U_1 \cap D_1$. In this manner we can define, for each $i > 1$, a nonempty set U_i, open in X, whose closure is countably compact and contained in $U_{i-1} \cap D_{i-1}$. Now $\{\bar{U}_i: i \geqslant 2\}$ is a decreasing sequence of nonempty closed subsets and, by Theorem 3.10.2 p. 258 of [2], its intersection is nonempty, $\bigcap_{i=1}^{\infty} \bar{U}_i \neq \emptyset$. Thus X is a Baire space, since $\bigcap_{i=1}^{\infty} \bar{U}_i = \bigcap_{i=1}^{\infty} U_i \subset U_1 \cap (\bigcap_{i=1}^{\infty} D_i)$ – compare Ex. 3.10.E(e) of [2], p. 271, see also [1]. The following Corollary 2 is a consequence of Theorems 1 and 2 and Lemma 1. Namely, we have:

COROLLARY 2. *Let X be a locally countably compact, quasi-regular space, Y be a first countable one and Z be metric. If f is a function on $X \times Y$ to Z, such that f_x is continuous for all $x \in X$ and f_y is quasi-continuous for all $y \in Y$, then the set of points of continuity of f is dense in $X \times \{p\}$, for all $p \in Y$.*

Added in proof. A stronger result in this direction may be found in author's *Continuity points in $\{x\} \times Y$*, Bull. Soc. Math. France 108 (1980).

References

[1] J. Colmez, *Sur les espaces précompacts*, C. R. Acad. Paris 234 (1952), pp. 1019–1021.

[2] R. Engelking, *General Topology*, Warszawa 1977.

[3] K. R. Gentry, H. B. Hoyle, III, *Somewhat continuous functions*, Czechoslovak Math. J. 21 (96), (1971), pp. 5–12.

[4] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. 19 (1932), pp. 184–197.

[5] N. F. G. Martin, *Quasi-continuous functions on product spaces*, Duke Math. J. 28 (1961), pp. 39–44.

[6] T. Neubrunn, *A generalized continuity and product spaces*, Math. Slov. 26 (1976), pp. 97–99.

[7] — *Generalized continuity and separate continuity* (to appear).

[8] J. C. Oxtoby, *Spaces that admit a category measure*, J. Reine Angew. Math. 205 (1961), pp. 156–170.

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