

Z. Piotrowski
Department of Mathematics and Computer Sciences
Youngstown State University
Youngstown, OH 44555, USA

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Separate and Joint Continuity II.

This is a continuation of my article [Pt]. Here, we pose some important open problems pertaining to separate versus joint continuity of functions defined on products of certain "nice" topological spaces.

In what follows let X , Y and Z be spaces and let a function $f: X \times Y \rightarrow Z$ be given. For every fixed $x \in X$, the function $f_x: Y \rightarrow Z$ defined by $f_x(y) = f(x, y)$, where $y \in Y$, is called an x -section of f . A y -section of f is defined similarly. We say that a function $f: X \times Y \rightarrow Z$ is *separately continuous* if f is continuous with respect to each variable while the other variable is fixed, i.e. if all of its x -sections f_x and y -sections f_y are continuous. Given a function $f: \prod_{i=1}^n X_i \rightarrow Z$; we shall denote that f is separately continuous by $f: \prod_{i=1}^n X_i \xrightarrow{\text{sc}} Z$. Throughout this paper all the considered spaces are assumed to be Hausdorff.

§1. W. Sierpinski [Si] proved that if $X = Y = \mathbb{R}$ then every separately continuous function $f: X \times Y \xrightarrow{\text{sc}} \mathbb{R}$ is uniquely determined by its values at the points of a dense subset D of the domain.

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This result is valid if the domain space is, roughly speaking, either:

- a) both X and Y are metric and either X or Y is Baire [see Mc], or
- b) if X is Baire and Y is second countable (see [GN] and [Co]).

Remark 1. It will be interesting to know "the size" (in various senses) and Borel class of the set D , in general case.

Remark 2. The "almost-continuity" condition for a function to be "uniquely determined by its values at the point of a dense subset D of the domain" also seems to be worthwhile of some deeper analysis, see for example [Ne].

Problem 1. Characterize \mathcal{X} 's such that Bierpinski theorem holds, \mathcal{Y} being compact.

§2. R. Kershner [Ke] showed that the set $D(f)$ of discontinuity points of any separately continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ has the dimension at most $n-2$. As we know, if X is separable metric, then $\text{ind } X = \text{Ind } X = \text{dim } X$, where ind , Ind and dim stand for the small inductive dimension, the large inductive dimension and the covering dimension, respectively.

Problem 2. Let us assume that $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ are "nice" normal spaces and let $f: \prod_{i=1}^n \mathcal{X}_i \rightarrow \mathbb{R}$. Must $\text{Ind } D(f) \leq n-2$ (or $\text{dim } D(f) \leq n-2$)? In particular, is this true if \mathcal{X}_i 's are compact?

Remark 3. It is worthwhile to know that there have been studies of "the size" (in various senses) of $D(f)$ for a separately continuous function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. In fact, G. C. Young, W. H. Young [YY] (see also [Pt] p. 296) showed that $D(f)$ may be large in sense of cardinality - may be *uncountable* in *every* rectangle contained in the unit square. T. Tolstoff [To] constructed a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ whose $D(f)$ has a *positive* Lebesgue measure (!) being large in measure-theoretical sense.

§3. Following [SR] a space X is called *Namioka* if for any compact space Y and any metric Z :

- (*) every separately continuous function $f: X \times Y \rightarrow Z$
 there is a dense G_δ set $A \subset X$ s.t. $A \times Y \subset C(f)$,
 where $C(f)$ stands for the set of points of (joint)
 continuity of f .

Remark 4. It has been shown [Ch] that a metric space Z in this definition can be replaced by the unit interval. However, an interesting question is how far can we go in relaxing the condition upon the range space Z (see an analogical problem for Blumberg spaces (compare §7), ([PS] and [BP])).

Remark 5. One cannot expect Z to be "too large" for if [Ch] p. 459 shown that even in the case when $X = Y = [-1, 1]$ (closed interval with Euclidean topology), there is a *compact* space Z namely $Z = C([-1, 1]^2, [-1, 1])$ equipped with the pointwise convergence topology,

so that (*) fails.

The following problem constitutes essentially Problem 944 I recorded in the New Scottish Book (Wrocław, Poland) in 1978.

Problem 3. *Let X be Namioka, Y be compact and let Z be any second countable, or more generally, a space having σ -disjoint base. Does (*) hold?*

Since it has been shown ([SR]) that all completely regular Namioka spaces are Baire and, obviously, in Baire spaces residual sets coincide with sets containing dense G_δ subsets, we can replace the condition "dense G_δ set A " in (*) by "residual set A " (for completely regular X 's).

§4. R. Kershner [Ke] characterized the set $D(f)$ of discontinuity points of $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, namely

Let $S \subset \mathbb{R}^2$. Then S is $D(f)$ of a certain function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ iff S is an F_σ contained in the product of two sets of first category.

This result has been generalized to compact metric spaces, see [BN].

Problem 4. *Characterize $D(f)$ for functions $f: X \times Y \rightarrow \mathbb{R}$, where X and Y are compact spaces.*

§5. It has been shown [SR] that all metric Baire spaces or separable Baire spaces are Namioka.

Problem 5. What is a natural class of spaces containing all metric and all separable spaces such that Namioka and Baire spaces coincide?

§6. In his remarkable paper [Na] I. Namioka asked (Remarks 1.3(b) p. 520) whether every - what we call now - Namioka space is Baire. The negative answer was provided by M. Talagrand [T2] see §7. In the same article the following spectacular problem was posed:

Problem 6. (M. Talagrand) Let X be Baire, Y be compact and let $f: X \times Y \rightarrow \mathbb{R}$. Is $\mathcal{C}(f) \neq \emptyset$?

Remark 6. If one assumes additionally in Problem 6 that Y is *first countable*, then the positive answer has been shown in [LP2] even for a larger class of functions $f: X \times Y \rightarrow \mathbb{R}$ namely, it is enough that all x -sections f_x are continuous (with the exception, possibly, of a first category set), and all its y -sections f_y are quasi-continuous (= inverse image of every open set in the range is contained between an open set and its closure in the domain space; such functions, as shown by S. Marcus, do not have to be Lebesgue measurable!) compare also [PW].

§7. Let us recall that a topological space X is called *Blumberg*¹⁾ if for every function $f: X \rightarrow \mathbb{R}$ there is a dense subset D of X such that f restricted to D is continuous (on D). It is known [BG] that for metric spaces:

¹⁾ In 1922 H. Blumberg showed that \mathbb{R} has the mentioned property.

X is Blumberg iff X is Baire (iff X is Namioka, see [SR]).

H. E. White, Jr. [Wh] proved that there is a Baire space that is not Blumberg. M. Talagrand [T2] has showed that there is an α -favorable space (hence Baire) which is not Namioka.

If X or Y is a metrizable space then every $f: X \times Y \rightarrow \mathbb{R}$ is the pointwise limit of a sequence of continuous functions¹⁾, we shall write then $f \in B_1(X \times Y)$. Consequently, if the pointwise compact subsets of $C(X)$ are metrizable, then every $f: X \times Y \rightarrow \mathbb{R}$ belong to $B_1(X \times Y)$, Y being compact²⁾. Very recently G. Vera [Ve] extended these results. Following him we will say that a topological space X is *Moran space* (see [Mo]) if every $f: X \times Y \rightarrow \mathbb{R}$ is in $B_1(X \times Y)$, Y being any compact space.

In view of §6, and the just presented material we have:

Problem 7. What are the relationships in the class of Baire spaces between Namioka, Blumberg, Moran, Sierpinski spaces (defined in Problem 1) and spaces X for which Talagrand's problem has a positive solution.

Remark 7. The question whether every Baire Moran space is Namioka was posed in by G. Vera [Ve] and has been answered, in positive, by him in his recent article "Vector-valued first Baire class functions".

¹⁾ See [Ru], compare [En] and further discussion in [Pt] p. 299.

²⁾ It happens, for example, if X is the support of some Borel measure and has a dense σ -compact subset [Ru].

§8. It is known ([CT], [B2]) that if Y is second countable, and M is metric, then:

(**) for every separately continuous function

$f: X \times Y \rightarrow M$ there exists a residual set

$A \subset X$ such that $A \times Y \subset C(f)$.

(i) Be aware of the fact, that if Y is *first countable* (even metric complete) and $M = \mathbb{R}$, then (**) does not have to be true even in the case if X is the closed unit interval $[0,1]$! - [B1] see [Pt], Ex. 6.14 p. 313.

(ii) Also, if the space Y is assumed only to have a *countable network*¹⁾, which implies that Y is hereditarily Lindelöf and hereditarily separable, then again (**) does not have to hold, (see [T1], Remark (b), p. 241, see also [LP1], comments following Example 1, p. 288); see also [Pt], Ex. 6.13 p. 311.

Following [LP1] we say that a space Y is *co-Namioka*²⁾ if for every Namioka space X condition (*) of §3 holds.

1) A family $\mathcal{N} = \{N_s\}_{s \in S}$ of subsets of a space X is called a *network* if for every $x \in X$ and for every neighborhood U of x , there is $s_0 \in S$, such that $x \in N_{s_0} \subset U$.

2) This term has been used independently by G. Debs in a different sense, namely to denote these Y 's, such that for any Baire space X (*) holds. The class of Debs' co-Namioka spaces, denoted usually by N^* , contains all Corson-compact spaces. Recently, R. Deville [De] showed that N^* contains all the compacts $[0, \Gamma)$ (Γ -an ordinal), and all scattered compact K 's such that $K(\Omega) = \emptyset$, where Ω is the first uncountable ordinal. He asked also whether N^* contain all scattered compact spaces.

Well, by the definition, compact spaces are co-Namioka. We have shown [LP1] Theorem, p. 289, that k_ω -spaces are co-Namioka rel (LC), LC denotes the class of locally compact spaces, that is; if X is any locally compact space, Y is a k_ω -space, then (*) of §3 is true.

Further, every locally compact σ -compact space is co-Namioka. It easily follows from [CT] and [B2] that all second countable spaces are co-Namioka.

The space Y of (i) serves as an example of a complete metric, locally compact space which is *not* co-Namioka.

Likewise, Y of (ii) illustrates that not all hereditarily Lindelöf and hereditarily separable spaces must be co-Namioka.

Problem 8. *Characterize co-Namioka spaces.*

§9. Although as yet the class of Namioka spaces has not been characterized (internally), there is a need for the determination of permanence properties of Namioka spaces. Some invariants have already been discovered in [HJT], however the following problem is still open.

Problem 9. (R. Hansell [H1]) *If X is closed-hereditarily Baire and Namioka, is every nonempty closed subspace of X Namioka? Are dense \mathcal{G}_δ subspaces of Namioka spaces Namioka? What other permanence properties Namioka spaces have?*

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References

- [BG] Bradford, J. C., Goffman, C., Metric spaces in which Blumberg's theorem holds, Proc. Amer. Math. Soc. 11(1960), 667-670, MR 26 #3832.
- [BN] Breckenridge, J. C., Nishiura, T., Partial continuity, quasi-continuity and Baire spaces. Bull. Inst. Math. Acad. Sinica 4(1976), no. 2, 191-203, MR 58 #24174.
- [B1] Brown, J. B., Oral communication, Fall 1983.
- [B2] _____, _____, Spring 1984.
- [BP] _____, Z. Piotrowski, Co-Blumberg spaces, Proc. Amer. Math. Soc. (1986), 683-688, MR 87b:54012.
- [De] DeVille, R., Convergence ponctuelle et uniforme sur un espace compact, Universite de Paris VI, preprint.
- [CT] Calbrix, J., Troallic, J. P., Applications séparément continues, C. R. Acad. Sci. Paris 288(1979), 647-648, MR 80 #54009.
- [Ch] Christensen, J.P.R., Joint continuity of separately continuous functions, Proc. Amer. Math. Soc. 82(1981), 455-461, MR 82 #54012.
- [Co] Comfort, W. W., Functions linearly continuous on a product of Baire spaces, Boll. Un. Mat. Ital., 20(1965), 128-134, MR 31, #3549.
- [En] Engelking, R., On Borel sets and B-measurable functions in metric spaces. Annales. Soc. Math. Polon. (Comm. Math.) 10(1967), 145-149, MR 35 #66.
- Goffman, C., see Bradford, J. C.
- [GN] Goffman, C., Neugebauer, C. J., Linearly continuous functions, Proc. Amer. Math. Soc., 12(1961), 997-998, MR 25 #151.
- [H1] Hansell, R. W., letter of 21 November 1986.
- [H2] _____, Sums, products and continuity of Borel maps in nonseparable metric spaces (preprint).
- [HJT] _____, Jayne, J. E., Talagrand, M., First class selectors for weakly upper semi-continuous multivalued maps

in Banach spaces, *J. Reine Angew. Mathematik* 361(1985), 201-220, MR 87m:54059a.

Jayne, J. E., see Hansell, R. W. [HJT].

[Ke] Kershner, R., The continuity of functions of many variables, *Trans. Amer. Math. Soc.* 53(1943), 83-100, MR 4, 153.

[LP1] Lee, J. P., Piotrowski, Z., A note on spaces related to Namioka spaces. *Bull. Austral. Math. Soc.* 31(1985), 285-292, MR 87b: 54009.

[LP2] _____, _____, Another note on Kempisty's generalized continuity, *Internat. J. Math. & Math. Soc.* 11(1988), 657-664.

[Mc] McCoy, R. A., Separately continuous functions, *Amer. Math. Monthly*, 85(1978), 199.

[Mo] Moran, W., Separate continuity and supports of measures *J. London Math. Soc.* 44(1969), 320-324, MR 38 #4642.

[Na] Namioka, I., Separate continuity and joint continuity, *Pacific J. Math.* 51(1974), 515-531, MR 51 # 6693.

Neugebauer, C. J., see Goffman, C., [GN].

[Ne] _____, A class of functions determined by dense sets, *Archiv der Mathematik* 23(1961), 206-209.

Nishiura, T., see Breckenridge, J. C., [BN].

Piotrowski, Z., see Brown, J. B., [BP].

_____, see Lee, J. P., [LP1] and [LP2].

[Pt] _____, Separate and joint continuity, *Real Analysis Exchange*, 11(1985-86), 293-322, MR 87i:01042.

[PS] _____, Szymański, A., Concerning Blumberg's theorem, *Houston J. Math.* 10(1984), 109-115, MR 85b: 54025.

[PW] _____, Wingler, E., On separately continuous functions (preprint).

[Ru] Rudin, W., Lebesgue first theorem, *Math. Analysis and Applications, Part B.* Edited by L. Nachbin, *Adv. in Math. Supplem. studies* 7B. Academic Press (1981), 741-747, MR 82k: 28006.

[SR] Saint Raymond, J., Jeux topologiques et espaces de Namioka, *Proc. Amer. Math. Soc.* 87(1983), 499-504, MR 83m #54060.

[Si] Sierpinski, W., Sur une propriété de fonctions de deux

variables reelles, continues par rapport à chacune de variables, Publ. Math. Univ. Belgrade vol. 1(1932), 125-128.

Szymanski, A., see Piotrowski, Z., [PS].

Talagrand, M., see Hansell, R. W. [HJT].

[T1] _____, Deux generalisations d'un théorème de I. Namioka, Pacific J. Math. 81(1979), 239-251, MR 80k #54018.

[T2] _____, Espaces de Baire et espaces de Namioka, Math. Ann. 270(1985), 159-164, MR 86a: 54057.

[To] Tolstoff, T., On partial derivatives, Izv. Akad. Nauk SSSR, Ser. Mat 13(1949), 425-446, Engl. trans. Amer. Math. Soc. Transl. 1.(1952), 55-83, MR 11, 167.

Troallic, J. P., see Calbrix, J., [CT].

[Ve] Vera, G., Baire measurability of separately continuous functions, Quart. J. Math. Oxford (2), 39(1988), 109-116.

[Wh] White Jr., H. E., Topological spaces in which Blumberg's theorem holds, Proc. Amer. Math Soc. 44(1974), 454-462, MR 49 #6130.

Wingler, E., see Piotrowski, Z., [PW].

[YY] Young, G. C., Young, W. H., Discontinuous functions continuous with respect to every straight line, Quart. J. Math. Oxford Ser. 41(1910), 87-93.

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