SEPARATE AND JOINT CONTINUITY IN BAIRE GROUPS

Zbigniew Piotrowski

Youngstown State University
Department of Mathematics and Statistics


A. Bouziad (Topology Appl. (1993)) showed that the multiplication is continuous in a semitopological Baire group which is a paracompact $p$-space.

He also recently showed (Proc. Amer. Math. Soc (1996)) that the multiplication is continuous in semitopological Čech-complete groups. We shall prove (Theorem 5) that if $X$ is a Baire, Moore semitopological group, then $X$ is paratopological.

I. Introduction.

Let us recall that a group $G$ is called semitopological (resp. paratopological) if $G$ is equipped with a topology in which the multiplication is separately (resp. jointly) continuous.

Further, a paratopological group $G$ is called topological if the inversion in $G$ is continuous.

This way we have two natural, very important questions:

(A) What is a topology on $G$ so that every semitopological group with this topology is paratopological?

(B) What is a topology on $G$ such that every paratopological group with this topology is topological?

Both questions (A) and (B) received a lot of attention in the past.

It was R. Ellis [10], [11] who showed that local compactness of $G$ is an answer to both (A) and (B).

Answering (A), A. Bouziad [2] showed that every semitopological Baire group which is a paracompact $p$-space, is paratopological. Also Čech-completeness answers (A) - see [3].

The main purpose of this work is to answer (A) for some other Baire groups.

The main result (Theorem 5) is a straight corollary from the previous Theorems of this paper.

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II. Some Topological results.

In this chapter we define some necessary notions used in the sequel, as well as prove Theorem 1. By a space we shall mean a topological space.

A function $f : X \rightarrow Y$ is called quasi-continuous at a point $x \in X$, where $x \in A$ and $f(x) \in H$ we have:

$$A \cap \text{Int} f^{-1}(H) \neq \phi.$$  

A function $f : X \rightarrow Y$ is called quasi-continuous if it is quasi-continuous at every point of $X$. This condition defined by V. Volterra [1], p. 95 has been applied into separate versus joint continuity problems [14] or [31]; recently [8], under the name “modified continuity” was used in a generalization of Michael’s Selection Theorem. Let us mention here that there are not Lebesgue measurable quasi-continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$, as well quasi-continuous functions can be arbitrary class of Baire, [22].

Further, given $X, Y$ and $Z$; a function $f : X \times Y \rightarrow Z$ is said to be quasi-continuous with respect to the variable $x \in (p, q) \in X \times Y$ if for every neighborhood $N$ of $f(p, q)$ and for every neighborhood $U \times V$ of $(p, q)$ there exists a neighborhood of $p$, $U^1 \subset U$ and a nonempty open $V^1 \subset V$ such that for all $(x, y) \in U^1 \times V^1$ we have $f(x, y) \in N$. If $f$ is quasi-continuous with respect to the variable $x$ at every point of its domain, we say that $f$ is quasi-continuous with respect to $x$. Let us recall

**Theorem 1.** [24] Let $X$ be first countable, $Y$ be a Baire space, and $Z$ be regular. If $f : X \times Y \rightarrow Z$ has all of its $x$-sections $f_x$ quasi-continuous and all of its $y$-sections $f_y$ continuous, then $f$ is quasi-continuous with respect to $x$.

**Remark 1.** Quasi-continuity of the sections $f_x$ in Theorem 1 cannot be weakened to the following one: inverse images of nonempty open subsets $V$ of $Y$, if nonempty, have nonempty interiors. The reader will easily construct such a function $f : I^2 \rightarrow \mathbb{R}$.

III. Generalized oscillation function and quasi-continuity.

First we need a couple of definitions. If $A \subset X$ and $U$ is a collection of subsets of $X$, then the star of $A$ with respect to $U$, denoted by $\text{st}(A, U) = \bigcup \{U \in U : U \cap A \neq \emptyset\}$.

For $x \in X$, we write $st(x, U)$ instead of $\text{st}(\{x\}, U)$.

A sequence $\{G_n\}$ of open covers of $X$ is called a development of $X$ if for each $x \in X$, the set $\{st(x, G_n) : n \in N\}$ is a base at $x$.

A developable space is a space which has a development. Every metric space is developable; take the family $\mathcal{B}_\varepsilon$ of balls of diameter less than $\varepsilon$ as a development.

A regular, developable space is called a Moore space.

Although some similar notions have been used earlier, see [6] or [17], the original idea of a generalized oscillation function $\Omega$ is due to A. Szymański (private communication), along with most of the comments preceding Lemma 1, see below.

If $f : X \rightarrow Y$ is a function and $\mathcal{P}_n$ is an open cover of $Y$, then we set: $\Omega_n(f, \mathcal{P}_n) = \{x \in X : \text{there is an open neighborhood } U \text{ of } x \text{ and a member } P_n \text{ of } \mathcal{P}_n \text{ such that } f(U) \subset P_n\}$.

If $Y$ is a metric space and $\mathcal{P}_\varepsilon$ is the family of all balls of diameter less than $\varepsilon$, then $\Omega_\varepsilon(f, \mathcal{P}_\varepsilon)$ is the set of all points, where the oscillation of $f$ is less than $\varepsilon$. 
Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
\sin \frac{1}{x}, & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases}$$

Let $\varepsilon = 1$, then $\Omega_\varepsilon(f, \mathcal{P}_\varepsilon) = \mathbb{R} - \{0\}$.

Remark 2. Let $\mathcal{P}_n$ be an open cover of $Y$. Then the set $\Omega_n(f, \mathcal{P}_n)$ is open.

Lemma 1. Let $X$ and $Y$ be spaces and let $\{P_n\}$ be a development for $Z$. If $f : X \times Y \to Z$ is quasi-continuous with respect to $x$, then $\Omega_n(f, P_n)$ is dense in $\{x\} \times Y$, for every $x \in X$ and arbitrary but fixed $n$.

Proof. Let $x_0 \in X$ be arbitrary and let $G$ be a nonempty open subset of $Y$. Further, let $y_0$ be an arbitrary element of $V$. and let $P_n$ be an element of the development $\{P_n\}$ such that $f(x_0, y_0) \in P_n$.

By quasi-continuity of $f$ with respect to $x$ at $(x_0, y_0)$, for every neighborhood $N$ of $f(x_0, y_0)$, and every neighborhood $U \times V$ of $(x_0, y_0)$, there is a neighborhood $U'$ of $x_0$ with $U' \subset U$ and a nonempty open $V' \subset V$ such that for all $(x, y) \in U' \times V'$ we have $f(x, y) \in N$.

Since the sets $N$ and $V$ are arbitrary, we may assume $N = P_n$ and $V = G$, as defined above.

Observe that, $U' \times V' \subset U \times G$ whereas $f(U' \times U') \subset P_n$. We have also $(U' \times V' \cap (\{x_0\} \times G)) = \{x_0\} \times V' \neq \phi$.

The above two facts imply the density of $\Omega_n(f, P_n)$ in $\{x_0\} \times Y$, $V'$ being a nonempty open subset of $G$, $n = 1, 2, \ldots$.

Now, define $\Omega(f, \{P_n\}) = \bigcap_{n=1}^{\infty} \Omega_n(f, P_n)$. We have the following

Lemma 2. Let $f : X \to Y$, where $Y$ is a developable space with a development $\{P_n\}$. Then $\Omega(f, \{P_n\}) \subset C(f)$.

Proof. Let $x \in \Omega(f, \{P_n\})$ and let $V$ be an open set containing $f(x)$. Since $Y$ is developable, there is an open countable base $\{V_n\}$ around $f(x_0)$.

Now, since $x \in \Omega(f, \{P_n\})$, for every $n \in N$, there is an index $N$ such that there is an open neighborhood $U_N$ of $x$ and a member $V_N$ of $\{V_n\}$, such that $f(U_N) \subset V_N \subset V$, which proves that $f$ is continuous at $x$.

Theorem 2. Let $X$ be a space, $Y$ be Baire, $Z$ be a developable space with a development $\{P_n\}$. If $f : X \times Y \to Z$ is quasi-continuous with respect to $x$, then for all $x \in X$, the set $C(f)$ is a dense $G_\delta$ subset of $\{x\} \times Y$.

Proof. Since $f$ is quasi-continuous with respect to $x$, it follows from Lemma 1, that the sets $\Omega(f, \{P_n\})$ are dense in $\{x\} \times Y$; by Remark 2 these sets are also open. So, $\Omega(f, \{P_n\})$ is the intersection of countably many open and dense subsets of $\{x\} \times Y$.

Now, we are done by Lemma 2, since every point of $\Omega(f, \{P_n\})$ is a continuity point of $f$.

Theorem 3 below is a straightforward corollary from Theorems 1 and 2.

Theorem 3. Let $X$ be first countable, $Y$ be Baire and let $Z$ be Moore. If $f : X \times Y \to Z$ has:

a) all sections $f_x$ quasi-continuous and;

b) all sections $f_y$ continuous, then
$C(f)$ is a dense $G_δ$ subset of $\{x\} \times Y$ for every $x \in X$.

The above theorem generalizes Theorem of [25] and Theorem 2 of [26], where $Z$ is assumed to be metric.

The following two examples show that the assumptions that $X$ is “first countable” and that $Z$ is “Moore” in Theorem 3, are in a sense indispensable.

**Example 2** [20] Let $Y$ and $Z$ denote the closed unit interval $I = [0, 1]$ and let $X$ be the space $C_p(I, I)$ of continuous functions from $I$ into $I$ equipped with the pointwise topology. Then $f : X \times Y \to Z$ given by $f(x, y) = x(y)$ is separately continuous which is discontinuous at every point of $X \times Y$.

It is worth noting that $C_p(I, I)$ is a Tychonoff space having a countable network [21] and as such it is hereditarily Lindelöf and hereditarily separable. $C_p(I, I)$ is not a Fréchet space and thus it is not first countable, see [19] and [7].

The forthcoming Example 3 shows the necessity that the range space $Z$ is Moore.

**Example 3.** [5] Let $X = Y = [-1, 1]$ and let $Z$ be the space of mappings from $[-1, 1]^2$ into $[-1, 1]$ equipped with the pointwise topology. Thus, $F(x, y)$ is a function of $(a, b) \in [-1, 1]^2$ given by:

$$F(x, y)(a, b) = \begin{cases} \frac{2(x-a)(y-b)}{(x-a)^2 + (y-b)^2}, & \text{if this quotient is defined} \\ 0, & \text{otherwise} \end{cases}$$

$F$ is separately continuous, but not (jointly) continuous at any point. The space $Z$ is a “large” compact Hausdorff space.

**IV. Joint continuity triples and spaces of JC-type.**

We shall start from a few definitions. Given classes $A, B$ and $C$ of topological spaces. Assume $X \in A, Y \in B$ and $Z \in C$. Following [27] we say that the ordered triple $(X, Y, Z)$ is a joint continuity triple, denoted by JC-triple, if:

$$(*) \text{ for every separately continuous function } f : X \times Y \to Z, \text{ the set } C(f) \neq \emptyset$$

It follows from Namioka’s theorem that (Čech-complete, $σ$-compact and locally compact, metric) is a JC-triple.

We shall identify some other JC-triples, see Proposition 1 Remark 3 below.

Let us mention that it is an open problem [30], whether (Baire, compact Hausdorff, $ℜ$) is a JC-triple.

Also, given a class $K$ of topological spaces and let $X \in K$.

We say that $X$ is of JC-type if $(*)$ above holds for $X = Y = Z$. The family of spaces of JC-type is nonempty. In fact, by Baire’s theorem the reals $ℜ$ is of JC-type.

Also, by Baire-Lebesgue-Kuratowski-Montgomery theorem, every complete metric space is of JC-type. Really, although Namioka-type theorem does not hold here – see J. B. Brown’s counterexample in [28], every separately continuous function is of first class of Baire. Now, since the product of complete metric spaces $X$ and $Y$ is complete metric, hence Baire, $f$ being pointwise discontinuous, has a dense $G_δ$ set $C(f)$.

The following Proposition 1 is a consequence of Theorem 3 in part III.

**Proposition 1.** Every Baire, Moore space is of JC-type.
Remark 3. When identifying spaces of JC-type the following results are of interest.

N.F.G. Martin [18] proved that if \( X \) is first countable, \( Y \) is Baire and \( Z \) is metric, then every separately continuous function is quasi-continuous. Now, assume \( X \times Y \) is Baire, since there are metric Baire spaces \( X \) and \( Y \) such that \( X \times Y \) is not Baire \([12]\). So, a quasi-continuous function \( f \) from the Baire space \( X \times Y \) into a metric space has a dense \( G_\delta \) set \( C(f) \) of points of continuity. Hence, as a consequence if \( X \) is a metric Baire space, such that \( X^2 \) is Baire, then such an \( X \) is of JC-type. However, this result is weaker than our Proposition 1, above.

Remark 4. Due to the fact that we are guaranteed of the presence of \( C(f) \) in sets of type \( \{x\} \times Y \), we do not have to assume that the domain \( X \times Y \) of \( f \) is Baire space.

Remark 5. T. Neubrunn \([22]\) improved the results of N. F. G. Martin, mentioned in Remark 3, by relaxing the assumption on \( Z \). Apparently, the regularity of \( Z \) suffices. Again we assume that \( X \times Y \) is Baire. Now, if \( Z \) is either metric or second countable, then \( f \) being a quasi-continuous function from a Baire domain has a dense \( G_\delta \) set \( C(f) \) of points of continuity \([22]\). However, assuming the metrizability or countability of a base for \( Z \) we lose the generality of Neubrunn’s assumptions.

So, a general question arises:

Assume \( X \) is a Baire space. What are “large\(^1\)” spaces \( Y \) so that every quasi-continuous function \( f : X \to Y \) has \( C(f) \neq \emptyset \)?

Answering this question we will show that the class of all spaces \( Y \) that have both open-hereditarily countable pseudo-base and are hereditarily Lindelöf is “too large”.

In fact, we have:

Example 4. Let \( X \) be the reals \( \mathbb{R} \) with the Euclidean topology and let \( Y \) be the reals with the Sorgenfrey topology and let \( f : X \to Y \) be the identity function, i.e. \( f(x) = x, x \in X \).

Observe that \( f \) is quasi-continuous, however the set \( C(f) = \emptyset \).

Let us now turn to semitopological groups, i.e., groups in which the multiplication is separately continuous.

The following result holds:

Theorem 4. Let \( X \) be a first countable JC-type semitopological group. Then \( X \) is paratopological.

Proof. Let \((x_0, y_0)\) be a continuity point of the multiplication and let \((x, y)\) be an arbitrary but fixed element of \( X \times X \). Also, assume \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \). We will show that \( \lim_{n \to \infty} x_n y_n = xy \). The multiplication function will be denoted by \( f \). Separate continuity implies:

(i) \( \lim_{n \to \infty} f(x_0 x^{-1}, x_n) = \lim_{n \to \infty} (x_0 x^{-1} \cdot x_n) = x_0 x^{-1} x = x_0 e = x_0 \).

Also,

(ii) \( \lim_{n \to \infty} f(y_n y^{-1} y_0) = \lim_{n \to \infty} (y_n \cdot y^{-1} y_0) = yy^{-1} y_0 = ey_0 = y_0 \).

Now, since \((x_0, y_0)\) is a continuity point of the multiplication it follows:

(iii) \( \forall \{(a_n, b_n)\} : (a_n, b_n) \lim_{n \to \infty} (x_0, y_0) \Rightarrow f(a_n, b_n) \lim_{n \to \infty} f(x_0, y_0) \). So, in particular, let \( a_n = x_0 x^{-1} x_n \) and let \( b_n = y_n y^{-1} y_0 \).

Observe that (i) and (ii):

\(^1\)(i.e., neither metrizable, nor having a countable base)
lim_{n \to \infty} a_n = \lim_{n \to \infty} (x_0 x^{-1} \cdot x_n) = x_0 \text{ and likewise } \lim_{n \to \infty} b_n = \\
lim_{n \to \infty} (y_n \cdot y^{-1} y_0) = y_0. \text{ So, by (iii) we get} \\
lim_{n \to \infty} f(a_n, b_n) = \lim_{n \to \infty} (x_0 x^{-1} x_n \cdot y_n y^{-1} y_0) = x_0 y_0 \\
(*) \lim_{n \to \infty} (x_0 x^{-1} x_n \cdot y_n y^{-1} y_0) = x_0 y_0 \\
So, \lim_{n \to \infty} x_n y_n = \lim_{n \to \infty} x_n y_n e = \lim_{n \to \infty} (x x^{-1}) x_n y_n (y^{-1} y) = \lim_{n \to \infty} (x x^{-1} x_n y_n (y^{-1} y_0) (y_0^{-1} y) = (x x^{-1} [\lim_{n \to \infty} (x_0 x^{-1} x_n y_n y^{-1} y_0)]) \\
(y_0^{-1} y) = x \cdot x_0^{-1} \cdot x_0 \cdot y_0 \cdot y_0^{-1} \cdot y = x \cdot (x_0^{-1} \cdot x_0) \cdot (y_0^{-1} y_0) = x e y = xy. \quad \square 

The main result of the paper, Theorem 5 below follows now trivially from Proposition 2 and Theorem 4.

**Theorem 5.** Let $X$ be a Baire, Moore semitopological group. Then $X$ is paratopological.

**References**

14. _____, *Quasi continuity and Namioka’s theorem*, Topology Appl. 46 (1992), 135-149.