Tatra Mt. Math. Publ. **42** (2009), 1–11

TATRA MOUNTAINS
Mathematical Publications

ON THE SEPARATELY OPEN TOPOLOGY

Zbigniew Piotrowski — Robert W. Vallin — Eric Wingler

ABSTRACT. We consider the relationship between separately continuous functions and separately open sets, and we study the properties of the separately open topology on $\mathbb{R}^2$ and on $\mathbb{Q}^2$. We show that $\mathbb{R}^2$ with this topology (denoted $\mathbb{R} \otimes \mathbb{R}$) is completely and strongly Hausdorff and that $\mathbb{Q} \otimes \mathbb{Q}$ is normal but not a p-space. In addition, we show that each point of $\mathbb{Q} \otimes \mathbb{Q}$ has an uncountable neighborhood base.

1. Introduction

This paper deals with two topologies on the space $\mathbb{R}^2$, the usual Euclidean topology and the separately open (or plus) topology. In this paper we will compare and contrast these topologies and the G_δ sets formed by each.

Let f be a function from $\mathbb{R}^2$ into $\mathbb{R}$. We say that f is continuous with respect to x (with respect to y) if the restricted function $f_y(x) = f(x, y)$, where y is fixed ($f_x(y) = f(x, y)$, where x is fixed) is a continuous function from $\mathbb{R}$ into $\mathbb{R}$. If f is continuous with respect to both x and y, then f is called a separately continuous function. The canonical example of a function that is separately continuous at a point where it is not continuous, is

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases} \qquad (*)$$

Since f is not continuous at $(0, 0)$, we know that there exist open intervals $I = (-a, a)$ such that $f^{-1}(I)$ is not an open Euclidean set in the plane. It is natural to ask what such a set $f^{-1}(I)$ looks like. The answer is a separately open set containing the origin.

Definition 1. The ε-*plus* at (a, b) of radius $\varepsilon > 0$ is

$$P_\varepsilon(a, b) = \left\{(x, b) \in \mathbb{R}^2 : |x - a| < \varepsilon\right\} \cup \left\{(a, y) \in \mathbb{R}^2 : |y - b| < \varepsilon\right\}.$$

(Note: We shall use $B_\varepsilon(a, b)$ to denote a Euclidean open ball about (a, b).)

2000 Mathematics Subject Classification: Primary 05C38, 15A15; Secondary 05A15, 15A18.

Keywords: separately open topology, separate continuity.

More generally, if X and Y are topological spaces, $(p,q) \in X \times Y$, and U and V are open neighborhoods of p and q, respectively, we define the (U,V)-*plus* at (p,q) by

$$P_{U,V}(p,q) = \{(x,q) : x \in U\} \cup \{(p,y) : y \in V\}.$$

Definition 2. A set $G \subset \mathbb{R}^2$ is said to be *separately open* if for each point $(a,b) \in G$ there exists $\varepsilon > 0$ such that $P_\varepsilon(a,b) \subset G$.

In general, the separately open topology is formed as follows: Let $X_1, X_2, \dots, X_n$ be a finite collection of topological spaces and let $X = \prod_{i=1}^n X_i$. We say that $S \subset X$ is separately open provided that for each $x = (x_1, x_2, \dots, x_n) \in S$ and each $i = 1, 2, \dots, n$ there is a neighborhood N_i of x_i in X_i such that $\prod_{i=1}^n A_i \subset S$ where $A_j = \{x_j\}$ when $j \neq i$ and $A_i = N_i$. For more information, see [6] and [7].

2. Structure of separately open sets

It is obvious that Euclidean open sets are separately open. The following example shows that the converse is not true.

Example 1. The Maltese Cross

$$A = \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : |y| > |3x|\} \cup \left\{(x,y) \in \mathbb{R}^2 : |y| < \left|\frac{x}{3}\right|\right\}$$

is a separately open, but not Euclidean open set.

The Maltese Cross has only one point $(0,0)$ where it is not open in the usual sense; that is, it is the union of an open set with a singleton. Obviously, one can quickly come up with a set with an infinite number of such points. For example, let

$$A_{(0,0)} = A \cap \left[\left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)\right],$$

and let $A_{(i,j)} = (i,j) + A_{(0,0)}$ for each $(i,j) \in \mathbb{Z}^2$. Then $\cup\{A_{(i,j)} : (i,j) \in \mathbb{Z}^2\}$ is separately open, but each point $(i,j) \in \mathbb{Z}^2$ lies outside of the (Euclidean) interior.

Example 2. Another example of a separately open set that is not Euclidean open was given by Popvassilev [12]. Remove any circle from the plane letting one point P of this circle remain. The remaining set is separately open, but P is not in the (Euclidean) interior.

These routine examples motivate us to ask the following question: Where can these points of "essential" separate openness occur; that is, can a nonempty separately open set be constructed in a way different from adding points to an existing nonempty open set?

The answer to this question is *yes.* We mention here a few ways to show this. One of the easiest examples to construct is the following: Let α and β be real numbers such that

$$\alpha^2 + \beta^2 = 1 \quad \text{and} \quad \frac{\alpha}{\beta} \notin \mathbb{Q},$$

and let f be the rotation defined by

$$f(x, y) = (\alpha x + \beta y, -\beta x + \alpha y).$$

Then it can be easily seen that the set $G = f(\mathbb{Q}^2)$ has the property that every horizontal or vertical line intersects it in at most one point. Hence $\mathbb{R}^2 \setminus G$ is separately open. Since G is dense in $\mathbb{R}^2$ under the usual topology, $\mathbb{R}^2 \setminus G$ cannot be obtained by adding points to an existing nonempty open set.

The following is a construction that can be generalized to other topological spaces. In the unit square $I \times I$, where $I = (0, 1)$, pick a countable base $\mathcal{B} = \{B_1, B_2, \dots\}$. Using induction, we shall first construct a dense countable set D that has at most one point in common with every horizontal and every vertical segment. (Such a set D is called a dense *thin* subset of $I \times I$, see [11].) First, choose an arbitrary point (x_1, y_1) of B_1. Suppose that for some natural number n we have already chosen (x_1, y_1), (x_2, y_2), …, (x_n, y_n) such that $(x_i, y_i) \in B_i$ and if $i \neq j$, then $x_i \neq x_j$ and $y_i \neq y_j$. Since every set in $\mathcal{B}$ is of cardinality $\mathfrak{c}$, by the Pigeonhole Principle we can pick $(x_{n+1}, y_{n+1}) \in B_{n+1}$ such that $x_{n+1} \neq x_i$ and $y_{n+1} \neq y_i$ for $i = 1, 2, \dots, n$. Let $D = \{(x_n, y_n) : n \in \mathbb{N}\}$. By construction, the set D is countable and dense. Now, let $G = (I \times I) \setminus D$. It is easy to see that G is separately open. The above construction can be generalized to fairly general topological spaces, e.g., both spaces in the product being Baire spaces having countable π-weight. (For results on thin and very thin dense sets, see [16], [13], and [5].)

Finally, Hart and Kunen [6, Remark 2.2] give the following example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 1–1 function whose graph Γ is dense in the plane. Then $\mathbb{R}^2 \setminus \Gamma$ is separately open. However, since Γ is dense in the plane, $\mathbb{R}^2 \setminus \Gamma$ has an empty interior, so it cannot be derived by adding points to a nonempty open set.

Example 3. The Maltese Cross A is a G_δ set in the Euclidean topology. If we let $A_n = A \cup B_{1/n}(0, 0)$, then each A_n is Euclidean open and $\cap A_n = A$.

Theorem 1. *If C is a separately open subset of $\mathbb{R}^2$ and is Euclidean open at all points except those in a set $E \subset C$ that is a G_δ set in the Euclidean topology, then C is a G_δ set in the Euclidean topology.*

Proof. Since E is a G_δ set, there is a countable collection of Euclidean open sets U_n such that $E = \cap U_n$. The set $C_n = C \cup U_n$ is Euclidean open for each n and $C = \cap C_n$. □

Question. The set E can be finite or in some cases countably infinite, but how far can we extend this exceptional set? Will any countable set do? What about a nowhere dense set?

It is not the case, though, that every separately open set is a Euclidean G_δ one.

EXAMPLE 4. Let $S = \{(x,x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ and let $G = \mathbb{R}^2 \setminus S$. Then each x-section and each y-section is open in $\mathbb{R}$ so G is separately open. However, G is not a Euclidean G_δ set because if it were, then $G \cap \{(x,x) : x \in \mathbb{R}\}$ would be a G_δ subset of the line $y = x$. This is impossible since this set is homeomorphic to $\mathbb{Q}$.

This example shows that it is not sufficient for the set E in Theorem 1 to be countable. We note that in this example the set $\mathbb{R} \setminus \mathbb{Q}$ could be replaced by any other subset of $\mathbb{R}$ (G_δ or not, nor even Borel) and the resulting set G would be separately open. This shows that the cardinality of the collection of separately open sets in $\mathbb{R}^2$ is $2^{\mathfrak{c}}$, and since the cardinality of the collection of Borel subsets of $\mathbb{R}^2$ is $\mathfrak{c}$, there must exist separately open sets that are not Borel sets.

An interesting fact about the usual topology on $\mathbb{R}^2$ is that each open set can be expressed as the inverse image of an open set in $\mathbb{R}$ under some continuous function. In particular, if $G \subset \mathbb{R}^2$ is open and $f(x)$ is the distance from x to $\mathbb{R}^2 \setminus G$, then $f^{-1}\big((0,\infty)\big) = G$.

Question. Is every separately open set in $\mathbb{R}^2$ the inverse image of an open set in $\mathbb{R}$ under a separately continuous function?

The answer to this question is *no,* as can be seen by the following cardinality argument. The cardinality of $\{G \subset \mathbb{R} : G \text{ is open}\}$ is $\mathfrak{c}$, and since a separately continuous function on $\mathbb{R}^2$ is uniquely determined by its values on a dense subset (such as $\mathbb{Q}^2$) of $\mathbb{R}^2$ (see [14]), the cardinality of the set of separately continuous functions is $\mathfrak{c}$. Hence the cardinality of

$$\big\{f^{-1}(G) : G \text{ is open in } \mathbb{R} \quad \text{and} \quad f\colon \mathbb{R}^2 \to \mathbb{R} \text{ is separately continuous}\big\}$$

is $\mathfrak{c}$. However, the cardinality of the collection of all separately open sets in $\mathbb{R}^2$ is $2^{\mathfrak{c}}$. It follows that most separately open sets in $\mathbb{R}^2$ cannot be expressed as the inverse image of an open set in $\mathbb{R}$ under a separately continuous function.

3. Generalized separate oscillation

In this section we will assume that all spaces are Hausdorff.

Let Z be a topological space. A sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of Z is called a *development* of Z if for each $z \in Z$ the set $\{st(z, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at z. A regular developable space is called a *Moore space*.

Further, a completely regular space Z is a *p-space* if there exists a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of families of open subsets of βZ such that

(1) each $\mathcal{G}_n$ covers Z;

(2) for each $z \in Z$, $\cap\{st(z, \mathcal{G}_n) : n \in \mathbb{N}\} \subset Z$.

It is known that every Čech-complete is a p-space.

The following term was introduced in [8]:

Definition 3. We say that a topological space Z has the property (*) if there is a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of Z such that if $z \in G_n \in \mathcal{G}_n$ for each n, and if W is an open set in Z that contains z, then $\cap\{G_j : 1 \leq j \leq n\} \subset W$ for some n.

In the class of completely regular spaces, p-spaces with a G_δ-diagonal coincide with spaces that have property (*). Also, every developable space has the property (*). (See [8] for additional information.)

Refining the generalized oscillation ω_f introduced in [8], we will now define a generalized separate oscillation ω_f^{sep} of $f\colon X \times Y \to Z$. Define the *generalized separate oscillation* ω_f^{sep} of f on the (U, V)-plus $P = P_{U,V}(p, q)$ by

$$\omega_f^{sep}(P) = \inf\left\{\frac{1}{n} : n \in \mathbb{N},\ \exists\, G \in \mathcal{G}_n \text{ such that } f(P) \subset G\right\}.$$

The *generalized oscillation* ω_f^{sep} of f is defined by

$$\omega_f^{sep}(p, q) = \inf\Big\{\omega_f^{sep}(P) : P \in \mathcal{P}(p, q)\Big\},$$

where $\mathcal{P}(p, q)$ stands for the collection of all (U, V)-pluses at (p, q).

4. An extension theorem

It is well-known [10, p. 422] that if f is a continuous function defined on a subset A of a metric space X with values in a complete metric space Y, then there exists a continuous extension f^* of f to a G_δ subset A^* of X. This motivates us to look for an analogous result for *separately* continuous functions defined on subsets of the Cartesian plane $\mathbb{R}^2$. To begin, let A be a subset of $\mathbb{R}^2$, and let f be a real-valued separately continuous function defined on A; that is, the restrictions of f to each horizontal and vertical section of A are continuous. Call a point p a *weak plus-accumulation point* of A if p is an accumulation point of $P_1(p) \cap A$ in the usual topology on $\mathbb{R}^2$. Call a point $p = (x, y)$ a *plus-accumulation point* of A if p is an accumulation point of both $\big(\{x\} \times \mathbb{R}\big) \cap A$ and $\big(\mathbb{R} \times \{y\}\big) \cap A$ in the usual topology on $\mathbb{R}^2$. Let A^+ denote the set A together with all its plus-accumulation points. For each point p in A^+ define $\omega^+(f, p)$, the separate oscillation[1] of f at p, to be the oscillation considered only over pluses at p; that is,

$$\omega^+(f, p) = \lim_{r \to 0} \sup\big\{|f(q_1) - f(q_2)| : q_1,\, q_2 \in A \cap P_r(p)\big\}.$$

(Notice that if p is an isolated point of A, in the sense that $P_r(p) \cap A = \{p\}$ for some r, then $\omega^+(f, p) = 0$.) Let A^* be the set of points p in A^+ for which $\omega^+(f, p) = 0$. To each $p = (x_0, y_0)$ in A^* assign the sequence $\{p_n\}$ in A with $p_n \to p$ and $p_n = (x_0, y_n)$ or $p_n = (x_n, y_0)$. Since $\omega^+(f, p) = 0$, we have

$$\lim_{n \to \infty} \operatorname{diam}\Big(f\big(\{p_n, p_{n+1}, \dots\}\big)\Big) = 0.$$

So $\big\{f\,(p_n)\big\}$ is a Cauchy sequence whose limit we will denote as $f^*(p)$. Then f^* is the extension of f to A^*. Separate continuity follows directly from the fact that $\omega^+(f^*, p) = 0$. Hence we have the following:

Theorem 2. *Let $f\colon A \to \mathbb{R}$ be a separately continuous function where $A \subset \mathbb{R}^2$ and let A^* be as defined above. If A is a proper subset of A^*, then f has a separately continuous extension to A^*.*

Remark 1. The statement of this theorem is far weaker than we would have liked, which would have been to say that A^* is a separately G_δ set (that is, the intersection of a countable collection of separately open sets). While it is true that the set of points in A^+ where $\omega^+(f^*,\, p) = 0$ is the intersection of the sets

$$A_n = \left\{p \in A^+ : \omega^+(f^*,\, p) < \frac{1}{n}\right\},$$

we cannot say that the sets A_n are separately open. For suppose $p \in A_n$. To show that A_n is open, we would need to show that there is $r > 0$ such that $P_r(p) \cap A^+$ is contained in A_n. However, for any $r > 0$ there may exist points q in $P_r(p) \cap A^+$ such that $\omega^+(f^*, q) \geq 1/n$, simply because there are points from A that lie on a plus centered at q that do not lie on a plus centered at p.

Even if all of the sets A_n were separately open, we still would not be able to say that f could be extended to a separately G_δ set, because it is not clear that A^+ is a separately G_δ set. While it is true that every horizontal and vertical

[1] Separate oscillation and generalized separate oscillation (mentioned in the previous section) are related. However, the former is an extended real-valued function, while the latter is bounded above by 1.

section of a separately G_δ subset of $\mathbb{R}^2$ is a G_δ subset of $\mathbb{R}$, the following question remains: If every horizontal and vertical section of a subset A of $\mathbb{R}^2$ is a G_δ subset of $\mathbb{R}$, is A a separately G_δ set?

Upon examining the proof of the preceding theorem, one might think that instead of using plus accumulation points in the definition of A^+, we could have used weak plus accumulation points instead. The following example will show that this is not always possible.

Example 5. Let

$$f(p) = \begin{cases} 1 & \text{if } p \in \mathbb{Q} \times \big(\mathbb{Q} \setminus \{0\}\big); \\ 0 & \text{if } p \in (\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}), \end{cases}$$

and let

$$A = \Big[\mathbb{Q} \times \big(\mathbb{Q} \setminus \{0\}\big)\Big] \cup \big[(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})\big].$$

Then f is separately continuous on A, because it is constant on each horizontal and each vertical section. Each point $p = (x, 0)$ is a weak plus-accumulation point of A, and $\omega^+(p) = 0$. However, $f^*(p)$ will be either 0 or 1 depending on whether x is irrational or rational. Hence, f^* is not separately continuous on A^*.

The next example demonstrates a limitation on the above method used to obtain an extension.

Example 6. Let A be a countable dense subset of $(0,1)^2$ having at most one point in common with each horizontal and each vertical line. (The construction of such a set D is demonstrated in the text following Example 2.) Also, let B and C be disjoint subsets of A such that both B and C are dense in A and $A = B \cup C$. Now, consider the following two functions:

(1) $f_1 \colon A \to \mathbb{R}$, defined by $f_1(p) = 1$ for each $p \in A$, and
(2) $f_2 \colon A \to \mathbb{R}$, defined by $f_2(p) = 1$ for each $p \in B$ and $f_2(p) = -1$ for each $p \in C$.

Note that the extension function f_1^* is given by $f_1^*(p) = 1$ for each $p \in (0,1)^2$, but f_1^* cannot be obtained by the "sequence techniques" used above, because A has no plus-accumulation points. For the same reason, our technique does not extend f_2 continuously either.

The authors are grateful to the referee for supplying the previous example.

For abstract topological spaces, a corresponding result is Theorem 1.1 of [2].

5. Separation axioms

In this section we will discuss which separation axioms the plus topology satisfies. To distinguish between the space $X \times Y$ with the product topology and the space $X \times Y$ with the plus topology, we will denote the latter by $X \otimes Y$.

Henriksen and Woods [7] have shown that if each of X and Y has a countable π-weight and Y is a Baire space, then $X \otimes Y$ is not regular. A more explicit construction showing that $\mathbb{R} \otimes \mathbb{R}$ is not regular is provided by Hart and Kunen [6], where it is shown that if $D \subset \mathbb{R} \times \mathbb{R}$ is dense in the Tychonoff topology and can be viewed as the graph of a 1–1 function that is closed and discrete in the plus topology, then the non-regularity of $\mathbb{R} \otimes \mathbb{R}$ follows from Sierpinski's theorem (see [6]), which asserts that every such separately open set is dense in the plus topology. Yet another construction showing the non-regularity of $\mathbb{R} \otimes \mathbb{R}$, based on the Baire Category theorem, was provided by Popvassilev [12].

The space $\mathbb{R} \otimes \mathbb{R}$ is clearly Hausdorff because its topology is stronger than the usual topology, which is Hausdorff. More generally, it is shown in [6] that $X \otimes Y$ is Hausdorff if and only if both X and Y are Hausdorff.

Similar arguments can be made for the properties Urysohn, completely Hausdorff, and strongly Hausdorff. A space X is *Urysohn* (see [15]) if for each pair of distinct points x and y in X there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$. The space $\mathbb{R} \otimes \mathbb{R}$ is Urysohn because $\mathbb{R}^2$ is Urysohn, and a continuous function on $\mathbb{R}^2$ is also continuous on $\mathbb{R} \otimes \mathbb{R}$.

A space X is *completely Hausdorff* (see [15]) if for each pair of distinct points x and y there exist disjoint open sets U and V such that $x \in U$, $y \in V$, and $\overline{U} \cap \overline{V} = \varnothing$. If X is a Urysohn space, then it is completely Hausdorff. Hence, $\mathbb{R} \otimes \mathbb{R}$ is completely Hausdorff.

A Hausdorff space X is *strongly Hausdorff* (see [9]) if every infinite subset of X contains a sequence $\{x_n\}$ such that the terms x_n have pairwise disjoint neighborhoods in X. Again, since $\mathbb{R}^2$ is strongly Hausdorff and the plus topology is stronger than the usual topology, $\mathbb{R} \otimes \mathbb{R}$ is strongly Hausdorff as well; that is, a collection of pairwise disjoint neighborhoods in $\mathbb{R}^2$ is also a collection of pairwise disjoint neighborhoods in $\mathbb{R} \otimes \mathbb{R}$.

6. Other topological properties

Theorem 3. *A neighborhood base for a point in $\mathbb{Q} \otimes \mathbb{Q}$ must be uncountable.*

Proof. Suppose that $\{B_n\}$ is a countable neighborhood base of the point (x, y) in $\mathbb{Q} \otimes \mathbb{Q}$. We will construct inductively an open set G containing (x, y) such that $B_n \not\subset G$ for each n. Let (x_1, y_1) be a point in $B_1 \setminus \{(x, y)\}$. Suppose that points (x_1, y_1), (x_2, y_2), ..., (x_{n-1}, y_{n-1}) different from (x, y) have been selected respectively from B_1, B_2, ..., B_{n-1} so that no two of these points lie on the same horizontal or vertical line. Since B_n contains a plus centered at (x, y), there is a point (x_n, y) in B_n with $x_n \neq x$ such that $x_n \neq x_i$ for all $i = 1, 2, \ldots, n-1$. Now B_n contains a plus centered at (x_n, y), so there is a point (x_n, y_n) in B_n with $y_n \neq y_i$ for all $i = 1, 2, \ldots, n-1$. Hence $(x_n, y_n) \in B_n \setminus \{(x, y)\}$ and (x_n, y_n) does not lie on any horizontal or vertical line containing (x_i, y_i) for any $i < n$. Now, $G = \mathbb{Q}^2 \setminus \{(x_n, y_n) : n \in \mathbb{N}\}$ is an open set and $B_n \not\subset G$ for each $n \in \mathbb{N}$. Hence, a neighborhood base of (x, y) cannot be countable. □

Remark 2. Since there are at most $\mathfrak{c}$ subsets of $\mathbb{Q}^2$ and a neighborhood base of $\mathbb{Q} \otimes \mathbb{Q}$ must be uncountable, under the Continuum Hypothesis there must be exactly $\mathfrak{c}$ open neighborhoods of a point.

In view of the above construction, the cardinality of the neighborhood base of $\mathbb{R} \otimes \mathbb{R}$ must be uncountable. In fact, a neighborhood base for a point in $\mathbb{R} \otimes \mathbb{R}$ must have $2^{\mathfrak{c}}$ elements. This is an immediate corollary of the following theorem (see also [18, p. 739]).

Theorem 4 ([6, Lemma 2.1, p. 105]). *Suppose that X and Y are Hausdorff spaces, that $w(X) \leq \mathfrak{c}$, and that each non-empty open subset of X has size at least $\mathfrak{c}$. Suppose that there are disjoint countable sets $D_\alpha \subset Y$ for $\alpha < \mathfrak{c}$ such that each D_α is dense in Y. Then,*

$$\chi\big((p, q), X \otimes Y\big) \geq 2^{\mathfrak{c}} \qquad \textit{for all} \quad (p, q) \in X \times Y.$$

(For a discussion of the weight $w(X)$ of a topological space X and the character $\chi(p, X)$ of a point in X, see [3, pp. 27–28].) Note that our Theorem 3 does not imply nor is implied by this result.

Remark 3. A. V. Arhangel'skiĭ [1] introduced a class of spaces, called p-spaces, in the following way: X is called a p-space (cf. [4, p. 444]) if there exists a sequence $\{\mathcal{G}_n\}$ of open covers of X satisfying the following condition: For each $x \in X$ and each n, if G_n satisfies $x \in G_n \in \mathcal{G}_n$, then

(1) $\cap_n \overline{G}_n$ is compact, and
(2) $\{\cap_{i \leq n} \overline{G}_i : n \in \omega\}$ is an outer network for the set $\cap_n \overline{G}_n$; that is, every open set containing $\cap_n \overline{G}_n$ contains some $\cap_{i \leq n} \overline{G}_i$.

The class of p-spaces is rather large; it contains all metric spaces and all Čech-complete spaces. In the same article [1], Arhangel'skiĭ showed that if X is a p-space, $w(X) \leq \mathrm{card}(X)$ (see [9], Remark, p. 10).

Obviously, $\mathrm{card}(\mathbb{Q}^2) = \omega$, but we have just shown that $w(\mathbb{Q} \otimes \mathbb{Q})$ is uncountable. This proves that $\mathbb{Q} \otimes \mathbb{Q}$ is not a p-space.

Remark 4. It is natural to ask whether $\mathbb{Q} \otimes \mathbb{Q}$ is a regular space. In fact, it is. Recall (see [6]) that a σ-set is a separable metric space in which every F_σ set is also a G_δ set. Since every countable metric space (in particular, $\mathbb{Q}$ is a σ-set and $\mathbb{Q}$ is a countable non-discrete metric space, it follows from [6, Theorem 5.5, p. 118] that $\mathbb{Q} \otimes \mathbb{Q}$ is regular.

One of the cardinals used in set theory is the cardinal $\mathfrak{p}$ (see [17, p. 115]). It is known [17, Theorem 3.1(a), p. 116] that $\mathfrak{p} \geq \omega_1$. It follows from [6, Corollary 5.8, p. 119] that $\mathbb{Q} \otimes \mathbb{Q}$ is normal and strongly 0-dimensional. (For a definition of strongly 0-dimensional, see [3, p. 443].) Of course, it would be nice to see an elementary proof of the normality of $\mathbb{Q} \otimes \mathbb{Q}$.

REFERENCES

[1] ARHANGEL'SKIĬ, A. V.: *On a class of spaces containing all metric spaces and all locally bicompact spaces*, Sov. Math. Dokl. **4** (1963), 1051–1055.

[2] BURKE, D.—LUTZER, D.—LEVI, S.: *Functional characterizations of certain p-spaces*, Topology Appl. **20** (1985), 161–165.

[3] ENGELKING, R.: *General Topology*, in: Monografie Matematyczne 60, PWN-Polish Scientific Publishers, Warszawa, 1977.

[4] GRUENHAGE, G.: *Generalized Metric Spaces*, in: Handbook of Set-Theoretic Topology (K. Kunen, J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984.

[5] GRUENHAGE, G.—NATKANIEC, T.—PIOTROWSKI, Z.: *On thin, very thin, and slim dense sets*, Topology Appl. **154** (2007), 817–833.

[6] HART, J. E.—KUNEN, K.: *On the regularity of the topology of separate continuity*, Topology Appl. **123** (2002), 103–123.

[7] HENRIKSEN, M.—WOODS, R. G.: *Separate versus joint continuity: a tale of four topologies*, Topology Appl. **97** (1999), 175–205.

[8] HOLÁ, Ľ.—PIOTROWSKI, Z.: *Continuity points of functions with values in generalized metric spaces* (preprint).

[9] JUHÁSZ, I.: *Cardinal Functions in Topology*, in: Math. Centre Tracts 34, Amsterdam, 1975.

[10] KURATOWSKI, K.: *Topology. Vol. I.* PWN-Polish Scientific Publishers, Warszawa, 1966.

[11] PIOTROWSKI, Z.: *Dense subsets of product spaces*, Questions Answers Gen. Topology **11** (1993), 135–13.

[12] POPVASSILEV, S.: *Non-regularity of some topologies on $\mathbb{R}^n$ stronger than the standard one*, Math. Pannon. **5** (1994) 105–110.

[13] SCHRÖDER, J.: *Impossible thin dense sets*, Questions Answers Gen. Topology **13** (1995), 93–96.

[14] SIERPÍNSKI, W.: *Sur une propriété de fonctions de deux variables réelles, continues par rapport à chacune de variables*, Publ. Math. Univ. Belgrade **1** (1932) 125–128.

[15] STEEN, L. A.—SEEBACH, J. A.: *Counterexamples in Topology*. Holt, Rinehart and Winston, New York, 1970.

[16] SZEPTYCKI, P. J.: *Dense subsets of product spaces*, Questions Answers Gen. Topology **13** (1995), 221–222.

[17] VAN DOUWEN, E. K.: *The Integers and Topology*, in:Handbook of Set-Theoretic Topology (K. Kunen, J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984.

[18] VELLEMAN, D. J.: *Multivariable calculus and the plus topology*, Amer. Math. Monthly **106** (1999), 733–740.

Received September 22, 2007

Zbigniew Piotrowski
Eric Wingler
Department of Mathematics
Youngstown State University
Youngstown, OH 44555
U.S.A.
E-mail: zpiotr@math.ysu.edu
wingler@math.ysu.edu

Robert W. Vallin
Department of Mathematics
Slippery Rock University of PA
Slippery Rock, PA 16057
U.S.A.
E-mail: robert.vallin@sru.edu