On Sierpiński's theorem on the determination of separately continuous functions

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Abstract. Consider the following statement:

(∗) Let X, Y, and Z be spaces and let $f : X \times Y \to Z$ be separately continuous. Then f is uniquely determined by its values on any dense subset of $X \times Y$.

W. Sierpiński proved that (∗) holds if $X = Y = Z = \mathbf{R}$, the set of real numbers. We provide a large class of spaces satisfying (∗). An example is given showing some limitations of this set-up.

I. Introduction.

A *space* here means a topological space. Also, unless explicitly stated, no separation axioms are assumed upon the spaces considered.

It is well known (see e.g. [En] p.59) that every continuous function f from a space X into a Hausdorff space Y is uniquely determined by its values on any dense subset of X; in other words, if two continuous functions agree on a dense subset of X, then they agree throughout X.

Borel functions or Baire functions of any class fail to have the above property. (Consider step functions.) Also, connected functions are not determined by their values on dense subsets. (Consider two "topological sine curves" having different values at the origin.)

The just mentioned examples show that the unique determination of a function by its values on a dense subset is quite a strong property which requires almost the full strength of continuity. It should be emphasized that when we speak of a function as being determined by its values on a dense set, there is always a certain collection of functions under consideration (for example, continuous functions). The values of that function on a dense set and its membership in the collection are what uniquely determine it. (See [Ng].)

Nevertheless, W. Sierpiński [Si] showed that any real-valued, separately continuous function on $\mathbf{R}^n$ is uniquely determined by its values on any dense subset the domain space. Sierpiński's result has been proven again in [Ma] and [To] and generalized by [GN] and [Co].

Our generalization (see Structural Lemma) is applicable to a larger class of spaces not considered by any of the previous authors. We shall also provide an example (Example 1 below) showing a limitation of Sierpiński's theorem.

II. Sierpiński's Theorem. Questions and Answers.

The following seven questions come naturally while considering possible extensions of Sierpiński's result.

(1) Pertaining the *size* of the domain, does the therorem hold for the *infinite* product $[0, 1]^{\aleph_0}$?

(2) Relating to the *type* of the spaces considered in the product, there are the following four cases:

(a) Is the assumption that at least one of the factors of the domain space is Baire necessary?
(b) Can the domain space be the product of two Baire spaces, at least one of which is second countable?
(c) Can the domain space be the product of a Baire space and a compact Hausdorff space?
(d) Can the domain space be the product of two compact Hausdorff spaces?

(3) Pertaining to the *range*, is the assumption that the range space Z is the set $\mathbf{R}$ of the real numbers necessary?

(4) Relating to the type of *all x-sections* f_x and *all y-sections* f_y: Assume that all x-sections and all y-sections are determined by their values on dense subspaces. Is the function $f : \mathbf{R}^2 \to \mathbf{R}$ uniquely determined by its values on any dense subset?

Here are some answers to the above questions.

Ad (1). Sierpiński's theorem fails for the infinite product $[0,1]^{\aleph_0}$. (See [Co], Remark 2.3, p. 133.)

Ad (2a). Yes, see [GN], p. 998, where it is stated that the theorem fails for the domain space $\mathbf{Q} \times \mathbf{Q}$, where $\mathbf{Q}$ is the space of rational numbers.

Ad (2b). Yes, see [Co], Corollary 2.1, p. 132, or [GN], Theorem 1, p. 997. R. A. McCoy [MC] announced another related result: Let X and Y be metric spaces, at least one of which is a Baire space, and let Z be a Hausdorff space. If $f : X \times Y \to Z$ is a separately continuous function, then f is uniquely determined by its values on a dense set.

Ad (2c). We do not know the answer to this question. (See Remark 2 later in the text.)

Ad (2d). The answer is yes. (See [Tr].)

Ad (3). No, see either [Co], [GN], or [MC], where $\mathbf{R}$ is replaced by spaces Z that are completely regular, regular Hausdorff, or Hausdorff, respectively .

Ad (4). No, the following is a counterexample.

Example 1. Let $I = [0,1]$ and let D be a dense set in I^2 such that for each $x \in I$ each of the sets $D \cap (\{x\} \times I)$ and $D \cap (I \times \{x\})$ contains at most one point. (Note: D can be countable or uncountable.) Now for each $\alpha \in I$ let

$$f_\alpha(x,y) = \begin{cases} \alpha, & \text{if } (x,y) \in I^2 - D, \\ \pi, & \text{if } (x,y) \in D. \end{cases}$$

Then for $\alpha \neq \beta$ we have that $f_{\alpha,x}(y) \neq f_{\beta,x}(y)$ except possibly for one value of y. Hence, the x-sections of these functions are determined on dense sets. Similarly, it can be seen that the y-sections of these functions are determined on dense sets.

Now we observe that for all $(x,y) \in D$ and $\alpha \neq \beta$ we have $f_\alpha(x,y) = f_\beta(x,y)$ even though $f_\alpha \neq f_\beta$. Hence, the collection $\{f_\alpha : \alpha \in I\}$ is not determined on dense sets. □

III. The Structural Lemma.

We shall start this section with a definition that will be used in the sequel. Namely, following Z. Frolík [Fr], we say that a function $f : X \to Y$ is *feebly continuous* if for every open nonempty set $V \subset Y$ the following holds:

$$f^{-1}(V) \neq \emptyset \Rightarrow Int f^{-1}(V) \neq \emptyset.$$

Feebly continuous functions were introduced in connection with the preservation of baireness under functions ([Fr]; [Du], p.256). Step functions and the topological sine curve with the value y_0 at 0, $-1 \leq y_0 \leq 1$, serve as examples of feebly continuous functions.

What can be easily seen is that feeble continuity, in general, does not allow a single point jump discontinuity. However, the following function $f : \mathbf{R} \to \mathbf{R}$ *is* feebly continuous: $f(x) = x$ if $x \neq 1$ and $x \neq -1$, $f(1) = -1$, and $f(-1) = 1$. Nevertheless, the following is true.

Lemma. Let Y be T_2. Further, let $x_0 \in X$, D be dense in X, and let $f : X \to Y$ be a function. If $f(D) = \{a\} \neq \{b\} = f(\{x_0\})$, then f is not feebly continuous.

Proof. Since Y is T_2, there are open sets V_a and V_b around a and b, respectively, such that $V_a \cap V_b = \emptyset$. Since $b \in f^{-1}(V_b)$, we have $f^{-1}(V_b) \neq \emptyset$. Now,

$$f^{-1}(V_a \cap V_b) = f^{-1}(V_a) \cap f^{-1}(V_b) = \emptyset. \qquad (*)$$

Observe that

$$f^{-1}(f(D)) \subset f^{-1}(V_a).$$

In view of $(*)$, this implies

$$f^{-1}(f(D)) \cap f^{-1}(V_b) = \emptyset.$$

So,

$$D \cap f^{-1}(V_b) = \emptyset.$$

But D is dense in X (!); that is, D meets every open, nonempty subset of X. Hence, $Int f^{-1}(V_b) = \emptyset$; in other words, f is not feebly continuous. □

We are now ready for the main result of the paper.

Structural Lemma. Assume that every separately continuous function from the product $X = X_1 \times X_2 \times \ldots \times X_n$ into a completely regular space Z is feebly continuous. Then any separately continuous function from X into Z is determined by its values on any dense subset of the domain.

Proof. Let l and k be abitrary, separately continuous functions from X into Z that agree on a dense subset $D \subset X$ and such that $l(x_0) \neq k(x_0)$ for some point $x_0 \in X$. Since Z is completely regular, there is a continuous function $c : Z \to [0, 1]$ such that

$$c(l(x_0)) = 0 \text{ and } c(k(x_0)) = 1.$$

Here $l(x_0)$ and $\{k(x_0)\}$ play the respective roles of "a point" and "a closed set not containing the point," mentioned in the definition of complete regularity. Clearly, both compositions $c \circ l$ and $c \circ k$ are separately continuous as compositions of a continuous function with a separately continuous one. Since $c \circ l$ and $c \circ k$ agree on D, it follows that the function $f = c \circ k - c \circ l$ is a separately continuous function into $[-1, 1]$ that is 0 on D and 1 at x_0. Now by the above lemma, f is *not* feebly continuous. This contradicts the hypothesis that *every* separately continuous function from the product X into Z is feebly continuous. □

IV. Applications of the Structural Lemma.

In order to apply our structural lemma to Sierpiński-type theorems, we need to know when separate continuity of $f : X \times Y \to Z$ implies feeble continuity.

Before looking at a few results of this type, let us recall that a function $f : X \to Y$ is termed *quasicontinuous* if for every $x_0 \in X$ and any open sets U and V, where $x_0 \in U$ and $f(x_0) \in V$, we have $Int f^{-1}(V) \cap U \neq \emptyset$. The notion of a quasicontinuous function was introduced by V. Volterra and is mentioned in the now-classical monograph by R. Baire [Ba] (p. 95), which considers the properties of separately continuous functions defined on the plane. It is true that every continuous function is quasicontinuous and every quasicontinuous function is feebly continuous. The converse implications, in general, do not hold.

Also, call a space *locally second countable* if every point of the space possesses a neighborhood satisfying the second countability axiom.

Proposition ([Ne], Theorem 2) Let X be a Baire space, Y be locally second countable and Z be regular. Let $f : X \times Y \to Z$ be such that all y-sections f_y are quasicontinuous and all x-sections f_x are quasicontinuous with the exception of a set of first category. Then f is quasicontinuous.

The following Sierpiński-type theorem is a corollary from the Structural Lemma and the above proposition.

Corollary Let X be a Baire space, Y be locally second countable, and Z be completely regular. Let $f : X \times Y \to Z$ be separately continuous. Then f is uniquely determined by its values on any dense subset of the domain.

Proof. By the proposition above such an f is quasicontinuous, hence feebly continuous. Now we apply the Structural Lemma to complete the proof. $\square$

V. Remarks and Comments.

Remark 1. The technique used in our lemma cannot be automatically transferred to the case when the function $f : \prod_{i=1}^{n} X_i \to Y$ is pointwise discontinuous, or even if it is of the first class of Baire. Easy examples (e.g., the Riemann function) show that first class functions do allow single point jump discontinuities.

Remark 2. The problem whether every separately continuous real-valued function defined on a product of a Baire space X and a compact Hausdorff space is determined by its values on a dense subspace is naturally linked to the still unsolved problem of M. Talagrand [Ta1].

Talagrand's problem: Let X be Baire, Y be compact Hausdorff, and let $f : X \times Y \to \mathbf{R}$ be separately continuous. Is the set $C(f)$ of points of continuity nonempty?

Talagrand's example of an α-favorable space that is not Namioka provides an example of a real-valued separately continuous function defined on a product of a Baire space and a compact Hausdorff space still having "many" points of continuity ([Ta2], see also [P1] and [PW]).

We have the following question: If X is a Baire space and Y is a compact Hausdorff space, is every separately continuous function $f : X \times Y \to \mathbf{R}$ feebly continuous?

A positive answer to the above question would solve both Talagrand's problem and our question (2c). Obviously, we could then use the Structural Lemma: in the case of

Talagrand's problem, the domain $X \times Y$ is a Baire space; f, being feebly continuous, has a nonempty set $C(f)$ of points of continuity.

Remark 3. The class of functions determined by dense sets has been studied also in [Ng].

Remark 4. Methods used in the proof of the Structural Lemma have been used already in [GN] (see footnote 2 and Theorem $1'$, both p.997) in a much weaker case when the separately continuous function f is quasi-continuous.

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