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# SOME REMARKS ON METRIC PRESERVING FUNCTIONS

**Abstract**

The purpose of this paper is to study a behavior of continuous metric preserving functions $f$ with $f'(0) = +\infty$. First we show, via a simple example, that it is possible that such a function has no finite derivatives at any point. Then in Example 2 we construct a nondecreasing, differentiable, metric preserving function having infinite derivative at least at the points $x = 2^{-n}$ for each natural number, n.

**Definition 1** *We call a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ metric preserving iff $f(d) : M \times M \to \mathbb{R}^+$ is a metric for every metric $d : M \times M \to \mathbb{R}^+$, where $(M, d)$ is an arbitrary metric space and $\mathbb{R}^+$ denotes the set of nonnegative reals. We denote by $\mathcal{M}$ the set of all metric preserving functions. (See [1].)*

In the paper [2] it is shown that each metric preserving function $f$ has a derivative (finite or infinite) at 0. Such functions $f$ with $f'(0) < +\infty$ are Lipschitz functions with Lipschitz constant $f'(0)$. (See Theorem 3 in [2].)

In contrast with the property we will construct a continuous metric preserving function which is nowhere differentiable. This function is a slight modification of Van der Waerden's continuous nowhere differentiable function. (See [4].)

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**Example 1** *Define $h : \mathbb{R}^+ \to \mathbb{R}^+$ as follows*

$$h(x) = \begin{cases} x & x \le \frac{1}{2} \\ \frac{1}{2} + |x - [x] - \frac{1}{2}| & x > \frac{1}{2} \end{cases}.$$

*(where $[a]$ denotes the integer part of $a$). Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ as follows*

$$f(x) = \sum_{n=1}^{\infty} 2^{-n}.h(2^n.x) \text{ for each } x \in \mathbb{R}^+.$$

*Then $f$ is continuous and nowhere differentiable. It is not difficult to verify that $f \in \mathcal{M}$.*

**Definition 2** *Let $a, b, c \in \mathbb{R}^+$. We call the triplet $(a, b, c)$ a triangle triplet iff $a \le b + c$, $b \le a + c$, and $c \le a + b$. (See [3].)*

The following assertion is a generalization of Proposition 2.16 of [1].

**Theorem 1** *Let $g$, $h \in \mathcal{M}$. Let $d > 0$ be such that $g(d) = h(d)$. Define $w : \mathbb{R}^+ \to \mathbb{R}^+$ as follows*

$$w(x) = \begin{cases} g(x) & x \in [0, d), \\ h(x) & x \in [d, \infty). \end{cases}$$

*Suppose that $g$ is nondecreasing and concave. Let*

$$\forall x, y \in [d, \infty) : |x - y| \le d \Rightarrow |h(x) - h(y)| \le g(|x - y|).$$

*Then $w \in \mathcal{M}$.*

PROOF. Let $a, b, c \in \mathbb{R}^+$, $a \le b \le c \le a + b$. We show that $(w(a), w(b), w(c))$ is a triangle triplet. We distinguish two non-trivial cases.

a) Suppose that $a, b \in [0, d)$, and $c \in [d, \infty)$. Evidently $w(a) \le w(b) \le w(b) + w(c)$. Since $|g(d) - f(c)| \le g(|c - d|)$, we obtain $w(b) = g(b) \le g(d) + [\, g(a) - g(c - d)] \le g(a) + h(c) = w(a) + w(c)$. Since $g$ is concave, we have $g(d) + g(a + b - d) \le g(a) + g(b)$, which yields $w(c) \le g(d) + g(c - d) \le g(d) + g(a + b - d) \le w(a) + w(b)$.

b) Suppose that $a \in [0, d)$, and $b, c \in [d, \infty)$. Since $(d, b, c)$ is a triangle triplet, we obtain $w(a) \le g(d) = h(d) \le h(b) + h(c) = w(b) + w(c)$. Since $|h(b) - h(c)| \le g(|b - c|)$, we have $w(b) \le g(c - b) + h(c) \le g(a) + h(c) = w(a) + w(c)$, and $w(c) \le g(c - b) + h(b) \le g(a) + h(b) = w(a) + w(b)$. □

The following example shows that there is a monotone continuous function $f \in \mathcal{M}$ such that in every neighborhood of $0$ there is $x_0 > 0$ such that $f'(x_0) = +\infty$.

**Example 2** *There is $f \in \mathcal{M}$ such that*
*(i) $f$ is continuous and nondecreasing,*
*(ii) $f'(x)$ exists for each $x \in \mathbb{R}^+$ (finite or infinite),*
*(iii) $f'(2^{-n}) = +\infty$ for each $n \in \mathbb{N}$.*

Define $g : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$g(x) = \begin{cases} \sqrt{2x - x^2} & x \in [0,1), \\ 1 & x \in [1,\infty). \end{cases}$$

Evidently $g$ is nondecreasing and concave. Define $h : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$h(x) = \begin{cases} 0 & x = 0, \\ 1 & x \in (0,1), \\ \frac{1}{2} \cdot [3 - g(2-x)] & x \in [1,2), \\ \frac{1}{2} \cdot [3 + g(x-2)] & x \in [2,\infty). \end{cases}$$

Since $\forall x > 0 : 1 \le h(x) \le 2$, by Proposition 1.3 of [1] we have $h \in \mathcal{M}$. We shall show that the assumptions of Theorem 1 are fulfiled. Let $x, y \in [1,\infty)$, $| x - y | \le 1$. We distinguish three cases.

a) Suppose that $1 \le x \le y < 2$. Since $2 - x = (2-y) + (y-x)$, we have $g(2-x) \le g(2-y) + g(y-x)$. Thus $|h(x) - h(y)| = \frac{1}{2} \cdot [g(2-x) - g(2-y)] \le \frac{1}{2} \cdot g(y-x) \le g(|x-y|)$.

b) Suppose that $1 \le x < 2 \le y$. Since $g$ is nondecreasing, we obtain $g(2-x) \le g(y-x)$ and $g(y-2) \le g(y-x)$. Therefore $|h(x) - h(y)| = \frac{1}{2} \cdot [g(2-x) + g(y-2)] \le \frac{1}{2} \cdot [g(y-x) + g(y-x)] = g(|x-y|)$.

c) Suppose that $2 \le x \le y$. Since $y - 2 = (y \quad x) + (x - 2)$, we have $g(y-2) \le g(y-x) + g(x-2)$. Thus $|h(x) - h(y)| = \frac{1}{2} \cdot [g(y-2) - g(x-2)] \le \frac{1}{2} \cdot g(y-x) \le g(|x-y|)$.

Define $w : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$w(x) = \begin{cases} g(x) & x \in [0,1), \\ h(x) & x \in [1,\infty). \end{cases}$$

By Theorem 1 we have $w \in \mathcal{M}$. It is not difficult to verify that

1. $w$ is continuous and nondecresing,

2. $w(x) \le 2$ for each $x \in \mathbb{R}^+$,

3. $w(x) = 2$ for each $x \geq 3$,
4. $w'(x)$ exists for each $x \in \mathbb{R}^+$ (finite or infinite) ,
5. $w'(2) = +\infty$.

Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ as follows

$$f(x) = \sum_{n=0}^{\infty} 2^{-n}.w(2^n.x) \text{ for each } x \in \mathbb{R}^+.$$

It is not difficult to verify that (i)-(iii) hold.

**Question 1** *It is possible to characterize the set* $f'^{-1}(+\infty)$ *?*

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