Some Remarks on Metric Preserving Functions

Abstract

The purpose of this paper is to study the behavior of continuous metric preserving functions $f$ with $f'(0) = +\infty$. First we show, via a simple example, that it is possible that such a function has no finite derivatives at any point. Then in Example 2 we construct a nondecreasing, differentiable, metric preserving function having infinite derivative at least at the points $x = 2^{-n}$ for each natural number, $n$.

Definition 1. We call a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ metric preserving iff $f(d) : M \times M \to \mathbb{R}^+$ is a metric for every metric $d : M \times M \to \mathbb{R}^+$, where $(M,d)$ is an arbitrary metric space and $\mathbb{R}^+$ denotes the set of nonnegative reals. We denote by $\mathcal{M}$ the set of all metric preserving functions. (See [1].)

In the paper [2] it is shown that each metric preserving function $f$ has a derivative (finite or infinite) at 0. Such functions $f$ with $f'(0) < +\infty$ are Lipschitz functions with Lipschitz constant $f'(0)$. (See Theorem 3 in [2].)

In contrast with the property we will construct a continuous metric preserving function which is nowhere differentiable. This function is a slight modification of Van der Waerden's continuous nowhere differentiable function. (See [4].)

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317
Example 1 Define \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
h(x) = \begin{cases} 
  x & x \leq \frac{1}{2} \\
  \frac{1}{2} + |x - [x] - \frac{1}{2}| & x > \frac{1}{2}
\end{cases}
\]

(where \([a]\) denotes the integer part of \(a\)). Define \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
f(x) = \sum_{n=1}^{\infty} 2^{-n} h(2^n x) \text{ for each } x \in \mathbb{R}^+.
\]

Then \( f \) is continuous and nowhere differentiable. It is not difficult to verify that \( f \in \mathcal{M} \).

Definition 2 Let \( a, b, c \in \mathbb{R}^+ \). We call the triplet \((a, b, c)\) a triangle triplet iff \( a \leq b + c, b \leq a + c, \) and \( c \leq a + b \). (See [3].)

The following assertion is a generalization of Proposition 2.16 of [1].

Theorem 1 Let \( g, h \in \mathcal{M} \). Let \( d > 0 \) be such that \( g(d) = h(d) \). Define \( w : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
w(x) = \begin{cases} 
  g(x) & x \in [0, d), \\
  h(x) & x \in [d, \infty).
\end{cases}
\]

Suppose that \( g \) is nondecreasing and concave. Let

\[
\forall x, y \in [d, \infty) : |x - y| \leq d \Rightarrow |h(x) - h(y)| \leq g(|x - y|).
\]

Then \( w \in \mathcal{M} \).

Proof. Let \( a, b, c \in \mathbb{R}^+, \) \( a \leq b \leq c \leq a + b \). We show that \((w(a), w(b), w(c))\) is a triangle triplet. We distinguish two non-trivial cases.

a) Suppose that \( a, b \in [0, d) \), and \( c \in [d, \infty) \). Evidently \( w(a) \leq w(b) \leq w(b) + w(c) \). Since \(|g(d) - f(c)| \leq g(|c - d|)\), we obtain \( w(b) = g(b) \leq g(d) + [g(a) - g(c - d)] \leq g(a) + h(c) = w(a) + w(c) \). Since \( g \) is concave, we have \( g(d) + g(a + b - d) \leq g(a) + g(b) \), which yields \( w(c) \leq g(d) + g(c - d) \leq g(d) + g(a + b - d) \leq w(a) + w(b) \).

b) Suppose that \( a \in [0, d) \), and \( b, c \in [d, \infty) \). Since \((d, b, c)\) is a triangle triplet, we obtain \( w(a) \leq g(d) = h(d) \leq h(b) + h(c) = w(b) + w(c) \). Since \(|h(b) - h(c)| \leq g(|b - c|)\), we have \( w(b) \leq g(c - b) + h(c) \leq g(a) + h(c) = w(a) + w(c) \), and \( w(c) \leq g(c - b) + h(b) \leq g(a) + h(b) = w(a) + w(b) \).

The following example shows that there is a monotone continuous function \( f \in \mathcal{M} \) such that in every neighborhood of \( 0 \) there is \( x_0 > 0 \) such that \( f'(x_0) = +\infty \).
Example 2 There is \( f \in M \) such that

(i) \( f \) is continuous and nondecreasing,
(ii) \( f'(x) \) exists for each \( x \in \mathbb{R}^+ \) (finite or infinite),
(iii) \( f'(2^{-n}) = +\infty \) for each \( n \in \mathbb{N} \).

Define \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
g(x) = \begin{cases} \sqrt{2x - x^2} & x \in [0, 1), \\ 1 & x \in [1, \infty). \end{cases}
\]

Evidently \( g \) is nondecreasing and concave. Define \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
h(x) = \begin{cases} 0 & x = 0, \\ 1 & x \in (0, 1), \\ \frac{1}{2} \cdot [3 - g(2 - x)] & x \in [1, 2), \\ \frac{1}{2} \cdot [3 + g(x - 2)] & x \in [2, \infty). \end{cases}
\]

Since \( \forall x > 0 : 1 \leq h(x) \leq 2 \), by Proposition 1.3 of [1] we have \( h \in M \). We shall show that the assumptions of Theorem 1 are fulfilled. Let \( x, y \in [1, \infty), |x - y| \leq 1 \). We distinguish three cases.

a) Suppose that \( 1 \leq x \leq y < 2 \). Since \( 2 - x = (2 - y) + (y - x) \), we have \( g(2 - x) \leq g(2 - y) + g(y - x) \). Thus \( |h(x) - h(y)| = \frac{1}{2} \cdot |g(2 - x) - g(2 - y)| \leq \frac{1}{2} \cdot g(y - x) \leq g(|x - y|) \).

b) Suppose that \( 1 \leq x < 2 \leq y \). Since \( g \) is nondecreasing, we obtain \( g(2 - x) \leq g(y - x) \) and \( g(y - 2) \leq g(y - x) \). Therefore \( |h(x) - h(y)| = \frac{1}{2} \cdot [g(2 - x) + g(y - 2)] \leq \frac{1}{2} \cdot [g(y - x) + g(y - x)] = g(|x - y|) \).

c) Suppose that \( 2 \leq x \leq y \). Since \( y - 2 = (y - x) + (x - 2) \), we have \( g(y - 2) \leq g(y - x) + g(x - 2) \). Thus \( |h(x) - h(y)| = |h(x) - h(2)| \leq \frac{1}{2} \cdot g(y - x) \leq g(|x - y|) \).

Define \( w : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
w(x) = \begin{cases} g(x) & x \in [0, 1), \\ h(x) & x \in [1, \infty). \end{cases}
\]

By Theorem 1 we have \( w \in M \). It is not difficult to verify that

1. \( w \) is continuous and nondecreasing,
2. \( w(x) \leq 2 \) for each \( x \in \mathbb{R}^+ \).
3. \( w(x) = 2 \) for each \( x \geq 3 \),

4. \( w'(x) \) exists for each \( x \in \mathbb{R}^+ \) (finite or infinite),

5. \( w'(2) = +\infty \).

Define \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) as follows

\[
f(x) = \sum_{n=0}^{\infty} 2^{-n} w(2^n x) \text{ for each } x \in \mathbb{R}^+.
\]

It is not difficult to verify that (i)-(iii) hold.

**Question 1** It is possible to characterize the set \( f'^{-1}(+\infty) \)?

**References**


