ON VOLterra SPACES

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In 1881, back in the Cro-Magnon days of—what is now called—Modern Analysis, V. Volterra [5] proved the following spectacular result, where $X = Y = \mathbb{R}$:

(*) Let $f : X \to Y$ be a function for which $C(f)$ and $D(f)$ are dense. Then there is no function $g : X \to Y$ such that $C(f) = D(g)$ and $D(f) = C(g)$.

In this statement and throughout this paper for any function $f : X \to Y$, the sets $C(f)$, respectively $D(f)$, denote the points of $X$ at which $f$ is continuous, respectively discontinuous. Much of the credit must be given to Vito Volterra for his ingenuous proof; keep in mind that both Cantor's set theory (cardinality arguments) as well as the studies of the oscillation of a function were in their infancies.

The fact that the denseness of both sets $C(f)$ and $D(f)$ cannot be omitted from (*) easily follows from

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$, if $x$ is rational and $f(x) = -x$, if $x$ is irrational. Then $C(f) = \{0\}$, and $D(f) = \mathbb{R} \setminus \{0\}$. Now, let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 1$, if $x \geq 0$ and $g(x) = 0$, if $x < 0$. Then, clearly $C(f) = D(g)$ and $D(f) = C(g)$.

In this note we shall strongly generalize Volterra’s result to non-metrizable domains and ranges of considered functions.

Generalizations of the notion of oscillation $\omega$ of a function to cases of non-metrizable ranges go back at least as early as 1964, see Isbell’s monograph [3].

Suitable modifications of $\omega$ have been given for uniform spaces [1] or developable spaces [4].

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1. The second named author wishes to express his appreciation to the Department of Mathematics and Statistics at the University of Auckland for their hospitality during his sabbatical stay.
Definition 2. If $A \subseteq X$ and $\mathcal{U}$ is a collection of subsets of $X$, then $$\text{st}(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$ 

Definition 3. A sequence $\{\mathcal{G}_n\}$ of open covers of $X$ is called a development of $X$ if for every $x \in X$ the set $$\{\text{st}(x, \mathcal{G}_n) : n \in \mathbb{N}\}$$ is a base at $x$. A space which has a development is called a developable space.

Remark 4. All metric spaces have developments.

Let $f : X \to Y$ be a function and $\mathcal{G}$ be an open cover for $Y$. Define $$\Omega(f; \mathcal{G}) = \{x \in X : \exists \text{ open } U \text{ containing } x \text{ and } \exists V \in \mathcal{G} \text{ with } f(U) \subset V\}.$$ 

The sets $\Omega(f; \mathcal{G})$ are obviously open, as the union of open sets. Note that $C(f) \subseteq \Omega(f; \mathcal{G}).$

Now, let $\{\mathcal{E}_n\}$ be a sequence of covers of $Y$. Define $$\Omega(f; \{\mathcal{E}_n\}) = \bigcap_{n=1}^{\infty} \Omega(f; \mathcal{E}_n).$$

Clearly $\Omega(f, \{\mathcal{E}_n\})$ are $G_\delta$ sets.

Proposition 5. Let $f : X \to Y$ be a function, where $Y$ is a developable space with a development $\{\mathcal{G}_n\}$. Then $C(f) = \bigcap_{n=1}^{\infty} \Omega(f, \mathcal{G}_n)$.

Proof. Let $x \in \Omega(f, \{\mathcal{G}_n\})$. Then for each $n \in \mathbb{N}$ there is an open neighbourhood $U_n$ of $x$ and $V_n \in \mathcal{G}_n$ such that $f(U_n) \subset V_n$. Because $\{\mathcal{G}_n\}$ is a development it follows that $\{V_n\}$ is a local base at $f(x)$. Suppose $W \subset Y$ is an open set containing $f(x)$. Then there is a set $V_n$ with $V_n \subset W$. Thus $f(U_n) \subset W$, so $f$ is continuous at $x$ and hence $x \in C(f)$.

Remark 6. In view of a previous observation $C(f)$ is a $G_\delta$ subset of $X$.

Definition 7. A space $X$ is called Baire if its nonempty open sets are of second category, or equivalently if any countable intersection of open and dense sets is dense.

Theorem 8. Let $X$ be a nonempty Baire space, $Y$ be a developable space with a development $\{\mathcal{G}_n\}$. Then $(*)$ holds.

Proof. Suppose that $f, g : X \to Y$ are two functions and that $C(f)$ and $D(f)$ are dense. Then
1. \( C(f) \cup D(f) = X \) and \( C(f) \cap D(f) = \emptyset \)

2. \( C(g) \cup D(g) = X \) and \( C(g) \cap D(g) = \emptyset. \)

If a function \( g \) were to exist which satisfies \( C(f) = D(g) \) and \( C(g) = D(f) \) then in view of (1) and (2) this implies :

3. \( C(f) \cap C(g) = \emptyset. \)

We shall prove that (3) cannot happen.

In fact, since \( C(f) \) and \( C(g) \) are the sets of points of continuity of \( f \) and \( g \), respectively, both \( C(f) \) and \( C(g) \) are dense in \( X \) and, by Proposition 5, are \( G_\delta \) sets. Thus, since \( X \) is Baire, \( C(f) \cap C(g) \) is a dense \( G_\delta \) so cannot be empty, contradicting (3).

Remark 9. Observe that we have used no separation axioms for \( X \), nor the density-in-itself of \( X \).

Corollary 10. Let \( X \) be Baire space and let \( Y \) be a metric space. Then (*) holds.

Corollary 11. Let \( X \) be locally compact Hausdorff space and let \( Y \) be a metric space. Then (*) holds.

Corollary 12. Let \( X \) be a complete metric space and let \( Y \) be a metric space. Then (*) holds.

Corollary 13 (V. Volterra). Let \( X = \mathbb{R} \) and let \( Y = \mathbb{R}. \) Then (*) holds.

The assumption that \( X \) is a Baire space in Theorem 8 cannot be dropped. In fact, we have the following:

Example 14. Let \( \mathbb{Q} \) denote the set of rational numbers. Let \( A \) and \( B \) be two dense, co–dense subsets of \( \mathbb{Q} \), \( A \cap B = \emptyset \) and \( A \cup B = \mathbb{Q} \) – in view of an old result of E. Hewitt such sets exist (\( \mathbb{Q} \) is a dense–in–itself, metric space – hence resolvable).

Let us enumerate the elements of \( A \) and \( B \), respectively, i.e., \( A = \{a_1, a_2, a_3, \ldots\} \), \( B = \{b_1, b_2, b_3, \ldots\} \). Now define functions, \( f : \mathbb{Q} \rightarrow \mathbb{R} \) and \( g : \mathbb{Q} \rightarrow \mathbb{R} \) as follows;
\[ f(x) = \begin{cases} \frac{1}{n} & \text{if } x = a_i, a_i \in A \\ 0 & \text{if } x \in B \end{cases}, \quad g(x) = \begin{cases} \frac{1}{n} & \text{if } x = b_i, b_i \in B \\ 0 & \text{if } x \in A \end{cases} \]

Clearly, \( D(f) = C(g) = A \) and \( C(f) = D(g) = B \).

**Definition 15.** We say that a space \( X \) is *Volterra* if for each pair \( f, g : X \to \mathbb{R} \) for which \( C(f) \) and \( C(g) \) are dense in \( X \), the set \( C(f) \cap C(g) \) is also dense in \( X \).

**Remark 16.** Example 14 shows that \( \mathbb{Q} \) is not Volterra. We have shown in the proof of Theorem 8 that every Baire space is Volterra.

**Definition 17.** We say that \( X \) is *strongly Volterra*, if for any developable space \( Y \) and for each pair of functions \( f, g : X \to Y \) for which \( C(f) \) and \( C(g) \) are dense in \( X \), the set \( C(f) \cap C(g) \) is also dense in \( X \).

We have shown in the proof of Theorem 8 that every Baire space is strongly Volterra.

Is every (strongly) Volterra space Baire?

In trying to answer the question in the affirmative we may try to proceed by contradiction. Suppose \( X \) is not Baire so there is an open set \( U \) in \( X \) with \( U = \bigcup N_n \) where each \( N_n \) is nowhere dense; we can also assume the \( N_n \) are mutually disjoint. If also

\[ U \subseteq \text{cl}(\bigcup N_{2n-1}) \cap \text{cl}(\bigcup N_{2n}) \quad \text{(**)} \]

then we may define \( f, g : X \to \mathbb{R} \) by

\[ f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \bigcup N_{2n-1} \\ 0 & \text{if } x \in (X - U) \cup (\bigcup N_{2n}) \end{cases}, \quad g(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \bigcup N_{2n} \\ 0 & \text{if } x \in (X - U) \cup (\bigcup N_{2n-1}) \end{cases}. \]

Then \( C(f) = (X - U) \cup (\bigcup N_{2n}) \) and \( C(g) = (X - U) \cup (\bigcup N_{2n-1}) \), each of which is dense in \( X \). However, \( C(f) \cap C(g) = X - U \) which is not dense in \( X \), so \( X \) is not Volterra.

Is there any interesting class of spaces for which we can be sure that (***) holds?

Here is another observation which may be relevant to our problem.
Proposition 18. If $X$ is any topological space, $Y$ a metric space and $f : X \to Y$ a function then $C(f)$ is a $G_\delta$ set. Conversely, if $X$ is a topological space having a dense co-dense set and $S$ is a $G_\delta$ subset of $X$ then there is a function $f : X \to \mathbb{R}$ for which $C(f) = S$.

Proof of the first part is well-known and therefore omitted.

Proof of the second part : Let $D, E \subset X$ be dense subsets with $D = X - E$. Because $S$ is $G_\delta$, there is a nested sequence $\langle U_n \rangle$ of open sets with $S = \cap_{n \in \mathbb{N}} U_n$.

Set $U_0 = X$ and define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in S \\ \frac{1}{2n + 1} & \text{if } x \in D \cap (U_n - U_{n+1}) \\ \frac{1}{2n + 2} & \text{if } x \in E \cap (U_n - U_{n+1}) \end{cases}$$

The sets $S, D \cap (U_n - U_{n+1})$ and $E \cap (U_n - U_{n+1})(n \geq 0)$ are mutually disjoint and cover $X$, so $f$ is well-defined.

$S \subset C(f)$ because if $x \in S$ and $\varepsilon > 0$ then there is $n$ with $\frac{1}{2n + 1} < \varepsilon$, and $U_n$ is an open set containing $x$. Now for each $y \in U_n, |f(y)| \leq \frac{1}{2n + 1}$ so

$$|f(x) - f(y)| < \varepsilon.$$ 

$C(f) \subset S$ because if $x \notin S$ then $x \in U_n - U_{n+1}$ for some $n$ so for each open $U \subset X$ with $x \in U, U \cap D \neq \emptyset \neq U \cap E$ and so there is $y \in U$ with $|f(x) - f(y)| \geq \frac{1}{2n + 2} - \frac{1}{2n + 3} = \frac{1}{(2n + 2)(2n + 3)}$, so $x \notin C(f)$.

Remark 19. The result constituting the second part of Proposition 18 is known, however, the proof that we offer is much simpler, than those found in the literature.

References


