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ON VOLTERRA SPACES

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In 1881, back in the Cro-Magnon days of –what is now called –Modern Analysis, V. Volterra [5] proved the following spectacular result, where $X = Y = \mathbf{R}$:

(*) *Let $f : X \to Y$ be a function for which $C(f)$ and $D(f)$ are dense. Then there is no function $g : X \to Y$ such that $C(f) = D(g)$ and $D(f) = C(g)$.*

In this statement and throughout this paper for any function $f : X \to Y$, the sets $C(f)$, respectively $D(f)$, denote the points of X at which f is continuous, respectively discontinuous. Much of the credit must be given to Vito Volterra for his ingenious proof; keep in mind that both Cantor's set theory (cardinality arguments) as well as the studies of the oscillation of a function were in their infancies.

The fact that the *denseness* of both sets $C(f)$ and $D(f)$ cannot be omitted from (*) easily follows from

Example 1. Let $f : \mathbf{R} \to \mathbf{R}$ be given by : $f(x) = x$, if x is rational and $f(x) = -x$, if x is irrational. Then $C(f) = \{0\}$, and $D(f) = \mathbf{R} \setminus \{0\}$. Now, let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = 1$, if $x \geq 0$ and $g(x) = 0$, if $x < 0$. Then, clearly $C(f) = D(g)$ and $D(f) = C(g)$.

In this note we shall strongly generalize Volterra's result to non-metrizable domains and ranges of considered functions.

Generalizations of the notion of oscillation ω of a function to cases of non-metrizable ranges go back at least as early as 1964, see Isbell's monograph [3].

Suitable modifications of ω have been given for uniform spaces [1] or developable spaces [4].

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Definition 2. If $A \subset X$ and $\mathcal{U}$ is a collection of subsets of X, then $st(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

Definition 3. A sequence $\{\mathcal{G}_n\}$ of open covers of X is called a *development of* X if for every $x \in X$ the set $\{st(x, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at x. A space which has a development is called a *developable space*.

Remark 4. All metric spaces have developments.

Let $f : X \to Y$ be a function and $\mathcal{C}$ be an open cover for Y. Define

$$\Omega(f;\mathcal{C}) = \{x \in X : \exists \text{ open } U \text{ containing } x \text{ and } \exists V \in \mathcal{C} \text{ with } f(U) \subset V\}.$$

The sets $\Omega(f;\mathcal{C})$ are obviously open, as the union of open sets. Note that $C(f) \subset \Omega(f, \mathcal{C})$.

Now, let $\{\mathcal{C}_n\}$ be a sequence of covers of Y. Define $\Omega(f; \{\mathcal{C}_n\}) = \bigcap_{n=1}^{\infty} \Omega(f; \mathcal{C}_n)$.

Clearly $\Omega(f, \{\mathcal{C}_n\})$ are G_δ sets.

Proposition 5. *Let $f : X \to Y$ be a function, where Y is a developable space with a development $\{\mathcal{G}_n\}$. Then $C(f) = \bigcap_{n=1}^{\infty} \Omega(f, \mathcal{G}_n)$.*

Proof. Let $x \in \Omega(f, \{\mathcal{G}_n\})$. Then for each $n \in \mathbb{N}$ there is an open neighbourhood U_n of x and $V_n \in \mathcal{G}_n$ such that $f(U_n) \subset V_n$. Because $\{\mathcal{G}_n\}$ is a development it follows that $\{V_n\}$ is a local base at $f(x)$. Suppose $W \subset Y$ is an open set containing $f(x)$. Then there is a set V_n with $V_n \subset W$. Thus $f(U_n) \subset W$, so f is continuous at x and hence $x \in C(f)$.

Remark 6. In view of a previous observation $C(f)$ is a G_δ subset of X.

Definition 7. A space X is called *Baire* if its nonempty open sets are of second category, or equivalently if any countable intersection of open and dense sets is dense.

Theorem 8. *Let X be a nonempty Baire space, Y be a developable space with a development $\{\mathcal{G}_n\}$. Then (*) holds.*

Proof. Suppose that $f, g : X \to Y$ are two functions and that $C(f)$ and $D(f)$ are dense. Then

1. $C(f) \cup D(f) = X$ and $C(f) \cap D(f) = \varnothing$
2. $C(g) \cup D(g) = X$ and $C(g) \cap D(g) = \varnothing$.

If a function g were to exist which satisfies $C(f) = D(g)$ and $C(g) = D(f)$ then in view of (1) and (2) this implies :

3. $C(f) \cap C(g) = \varnothing$.

We shall prove that (3) cannot happen.

In fact, since $C(f)$ and $C(g)$ are the sets of points of continuity of f and g, respectively, both $C(f)$ and $C(g)$ are dense in X and, by Proposition 5, are G_δ sets. Thus, since X is Baire, $C(f) \cap C(g)$ is a dense G_δ so cannot be empty, contradicting (3).

Remark 9. Observe that we have used no separation axioms for X, nor the density-in-itself of X.

Corollary 10. *Let X be Baire space and let Y be a metric space. Then (*) holds.*

Corollary 11. *Let X be locally compact Hausdorff space and let Y be a metric space. Then (*) holds.*

Corollary 12. *Let X be a complete metric space and let Y be a metric space. Then (*) holds.*

Corollary 13 *(V. Volterra). Let $X = \mathbf{R}$ and let $Y = \mathbf{R}$. Then (*) holds.*

The assumption that X is a Baire space in Theorem 8 cannot be dropped. In fact, we have the following :

Example 14. Let $\mathbf{Q}$ denote the set of rational numbers. Let A and B be two dense, co-dense subsets of $\mathbf{Q}$, $A \cap B = \varnothing$ and $A \cup B = \mathbf{Q}$ – in view of an old result of E. Hewitt such sets exist ($\mathbf{Q}$ is a dense-in-itself, metric space – hence resolvable).

Let us enumerate the elements of A and B, respectively, i.e., $A = \{a_1, a_2, a_3, \ldots\}$, $B = \{b_1, b_2, b_3, \ldots\}$. Now define functions, $f : \mathbf{Q} \to \mathbf{R}$ and $g : \mathbf{Q} \to \mathbf{R}$ as follows ;

$$f(x) = \begin{cases} \frac{1}{i}, & \text{if } x = a_i, a_i \in A \\ 0, & \text{if } x \in B \end{cases}, \quad g(x) = \begin{cases} \frac{1}{i}, & \text{if } x = b_i, b_i \in B \\ 0, & \text{if } x \in A \end{cases}$$

Clearly, $D(f) = C(g) = A$ and $C(f) = D(g) = B$.

Definition 15. We say that a space X is *Volterra* if for each pair $f, g : X \to \mathbf{R}$ for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is also dense in X.

Remark 16. Example 14 shows that $\mathbf{Q}$ is not Volterra. We have shown in the proof of Theorem 8 that every Baire space is Volterra.

Definition 17. We say that X is *strongly Volterra*, if for any developable space Y and for each pair of functions $f, g : X \to Y$ for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap C(g)$ is also dense in X.

We have shown in the proof of Theorem 8 that every Baire space is strongly Volterra.

Is every (strongly) Volterra space Baire?

In trying to answer the question in the affirmative we may try to proceed by contradiction. Suppose X is not Baire so there is an open set U in X with $U = \cup N_n$ where each N_n is nowhere dense; we can also assume the N_n are mutually disjoint. If also

$$U \subset cl(\cup N_{2n-1}) \cap cl(\cup N_{2n}) \qquad (**)$$

then we may define $f, g : X \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} \text{ if } x \in \cup N_{2n-1} \\ 0 \text{ if } x \in (X - U) \cup (\cup N_{2n}) \end{cases}, \quad g(x) = \begin{cases} \frac{1}{n} \text{ if } x \in \cup N_{2n} \\ 0 \text{ if } x \in (X - U) \cup (\cup N_{2n-1}) \end{cases}.$$

Then $C(f) = (X - U) \cup (\cup N_{2n})$ and $C(g) = (X - U) \cup (\cup N_{2n-1})$, each of which is dense in X. However, $C(f) \cap C(g) = X - U$ which is not dense in X, so X is not Volterra.

Is there any interesting class of spaces for which we can be sure that (**) holds? Here is another observation which may be relevant to our problem.

Proposition 18. *If X is any topological space, Y a metric space and $f : X \to Y$ a function then $C(f)$ is a G_δ set. Conversely, if X is a topological space having a dense co–dense set and S is a G_δ subset of X then there is a function $f : X \to \mathbf{R}$ for which $C(f) = S$.*

Proof of the first part is well–known and therefore omitted.

Proof of the second part : Let $D, E \subset X$ be dense subsets with $D = X - E$. Because S is G_δ, there is a nested sequence $\langle U_n \rangle$ of open sets with $S = \cap_{n \in \mathbf{N}} U_n$.

Set $U_0 = X$ and define $f : X \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0 \text{ if } x \in S \\ \frac{1}{2n+1} \text{ if } x \in D \cap (U_n - U_{n+1}) \\ \frac{1}{2n+2} \text{ if } x \in E \cap (U_n - U_{n+1}) \end{cases}$$

The sets $S, D \cap (U_n - U_{n+1})$ and $E \cap (U_n - U_{n+1}) (n \geq 0)$ are mutually disjoint and cover X, so f is well–defined.

$S \subset C(f)$ because if $x \in S$ and $\varepsilon > 0$ then there is n with $\frac{1}{2n+1} < \varepsilon$, and U_n is an open set containing x. Now for each $y \in U_n, |f(y)| \leq \frac{1}{2n+1}$ so $|f(x) - f(y)| < \varepsilon$.

$C(f) \subset S$ because if $x \notin S$ then $x \in U_n - U_{n+1}$ for some n so for each open $U \subset X$ with $x \in U, U \cap D \neq \varnothing \neq U \cap E$ and so there is $y \in U$ with $|f(x) - f(y)| \geq \frac{1}{2n+2} - \frac{1}{2n+3} = \frac{1}{(2n+2)(2n+3)}$, so $x \notin C(f)$.

Remark 19. The result constituting the second part of Proposition 18 is known, however, the proof that we offer is much simpler, than those found in the literature.

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